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Spectral properties of flipped Toeplitz matrices and computational applications

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ABSTRACT

We study the spectral properties of flipped Toeplitz matrices of the form $H_n(f) = Y_n T_n(f)$, where $T_n(f)$ is the $n \times n$ Toeplitz matrix generated by the function f and Y_n is the $n \times n$ exchange (or flip) matrix having 1 on the main anti-diagonal and 0 elsewhere. In particular, under suitable assumptions on f, we establish an alternating sign relationship between the eigenvalues of $H_n(f)$, the eigenvalues of $T_n(f)$, and the quasi-uniform samples of f. Moreover, after fine-tuning a few known theorems on Toeplitz matrices, we use them to provide localization results for the eigenvalues of $H_n(f)$. Our study is motivated by the convergence analysis of the minimal residual (MINRES) method for the solution of real non-symmetric Toeplitz linear systems of the form $T_n(f)\mathbf{x} = \mathbf{b}$ after pre-multiplication of both sides by Y_n, as suggested by Pestana and Wathen [26]. A selection of numerical experiments is provided to illustrate the theoretical results and to show how to use the spectral localizations for predicting the MINRES performance on linear systems with coefficient matrix $H_n(f)$.

1. Introduction

A matrix of the form

$$[f_{i-j}]_{i,j=1}^{n} = \begin{bmatrix} f_0 & f_{-1} & \cdots & \cdots & f_{-(n-1)} \\ f_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & f_{-1} \\ f_{n-1} & \cdots & \cdots & f_1 & f_0 \end{bmatrix}, \tag{1.1}$$

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whose entries are constant along each diagonal, is called a Toeplitz matrix. In the case where the entries f_k are the Fourier coefficients of a function $f : [-\pi,\pi] \to \mathbb{C}$ in $L^1([-\pi,\pi])$, i.e.,

$$f_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x, \qquad k \in \mathbb{Z},$$

the matrix (1.1) is denoted by $T_n(f)$ and is referred to as the nth Toeplitz matrix generated by f.

The efficient solution of a linear system with a coefficient matrix of the form $T_n(f)$ by means of Krylov subspace methods is a research topic that involved several researchers over time. The main efforts focused on the case where $T_n(f)$ is real symmetric positive definite, so that the conjugate gradient (CG) method can be applied as well as its preconditioned version. Whenever $T_n(f)$ is real symmetric but indefinite, an alternative to (preconditioned) CG is the (preconditioned) minimal residual (MINRES) method. A common feature of CG and MINRES is that their convergence bounds are formally identical and rely only on the eigenvalues of the system matrix, or on the eigenvalues of the preconditioned system matrix if preconditioning is applied; see [8, Sections 2.1.1 and 2.2] for CG and [8, Sections 6.1 and 6.2.4] for MINRES. In the case where the Fourier coefficients of f are real but $T_n(f)$ is not symmetric, Pestana and Wathen [26] suggested pre-multiplying $T_n(f)$ by the $n \times n$ exchange (or flip) matrix

$$Y_n = \begin{bmatrix} & & & & 1 \\ & & & 1 & \\ & & \iddots & & \\ & 1 & & & \\ 1 & & & & \end{bmatrix}. \tag{1.2}$$

In this way, the resulting flipped matrix $H_n(f) = Y_nT_n(f)$ is real symmetric and (preconditioned) MINRES can be applied. Solving a real non-symmetric Toeplitz linear system $T_n(f)\boldsymbol{x} = \boldsymbol{b}$ by (preconditioned) MINRES applied to the symmetrized linear system $H_n(f)\boldsymbol{x} = Y_n\boldsymbol{b}$ has some advantages over solving the original system through either direct methods or iterative methods for non-symmetric Toeplitz matrices. For a detailed discussion on this topic, we refer the reader to [26, Sections 1–2], where a comparison between the proposed symmetrization approach and other relevant Toeplitz solvers available in the literature is presented. We also emphasize that the flipping strategy along with the solution by (preconditioned) MINRES of the resulting symmetrized linear system, originally proposed in [26] for real non-symmetric Toeplitz matrices, is gaining more and more popularity over time. In particular, this approach has recently been used for real non-symmetric multilevel Toeplitz matrices [18,19,25] as well as for the numerical treatment of evolutionary partial differential equations [17–19,24], optimal control problems [12,16] and image deblurring [7,11]—where real non-symmetric (multilevel block) Toeplitz structures naturally arise—and it proved to be a viable alternative to other notable solvers for the related non-symmetric linear systems [20,21,32].

A main reason behind the interest in the spectral properties of flipped Toeplitz matrices such as $H_n(f)$ is precisely the convergence analysis of MINRES for symmetrized Toeplitz systems. In this regard, a precise asymptotic spectral distribution theorem for the sequence of flipped Toeplitz matrices $\{H_n(f)\}_n$ was established independently by Ferrari et al. [9] through techniques based on the notion of approximating classes of sequences [13, Chapter 5], and by Mazza and Pestana [22] through the theory of block generalized locally Toeplitz sequences [2]. The same type of study was later extended to flipped multilevel Toeplitz matrices in [10,23]. However, no localization result for the eigenvalues of $H_n(f)$ was provided so far in the literature, despite the importance of spectral localization in the convergence analysis of MINRES.

In this paper, based on classical results for Toeplitz matrices [4,5,13] and on recent results on the asymptotic spectral distribution of arbitrary sequences of matrices [1], we delve deeper into the spectral properties of $H_n(f)$ under suitable assumptions on the function f. In particular:

- We show that the eigenvalues of $H_n(f)$ can be subdivided into two subsets of cardinalities $\lceil n/2\rceil$ and $\lfloor n/2\rfloor$ such that
  - the $\lceil n/2\rceil$ eigenvalues belonging to the first subset coincide with $\lceil n/2\rceil$ eigenvalues of $T_n(f)$ and have an asymptotic distribution described by f;
  - the $\lfloor n/2\rfloor$ eigenvalues belonging to the second subset coincide with $\lfloor n/2\rfloor$ eigenvalues of $-T_n(f)$ and have an asymptotic distribution described by $-f$.
- As an extension of the previous result, we show that, for every n, the eigenvalues of $H_n(f)$ are given by the following alternating sign relationship:

  $$\lambda_i(H_n(f)) = (-1)^{i+1}\lambda_i(T_n(f)) = (-1)^{i+1}f(x_{i,n}) + \varepsilon_{i,n}, \qquad i = 1,\dots,n,$$

  where $\lambda_i(T_n(f))$, $i = 1,\dots,n$, are the eigenvalues of $T_n(f)$, $\{x_{i,n}\}_{i=1,\dots,n}$ is an asymptotically uniform grid (see Section 2.1), and $\max_{i=1,\dots,n}|\varepsilon_{i,n}| \to 0$ as $n \to \infty$; moreover, $\varepsilon_{i,n} = 0$ for every $i = 1,\dots,n$ if f has a finite number of local maximum/minimum points and discontinuities.
- After fine-tuning a few known theorems on the singular values of Toeplitz matrices, we use them and other related theorems to provide localization results for the eigenvalues of $H_n(f)$.

The paper is organized as follows. Section 2 contains preliminaries. Section 3 contains statements and proofs of our main spectral results for flipped Toeplitz matrices of the form $H_n(f)$. Section 4 contains numerical experiments that illustrate part of the main theorems and show how the obtained localizations for the eigenvalues of $H_n(f)$ can be used to predict the MINRES performance on linear systems with coefficient matrix $H_n(f)$. Section 5 contains final remarks.

2. Preliminaries

2.1. Notation and terminology

We denote by μ the Lebesgue measure in $\mathbb{R}$. Throughout this paper, all terminology from measure theory (such as "measurable", "almost everywhere (a.e.)", etc.) always refers to the Lebesgue measure. The closure of a set E is denoted by $\overline{E}$. We use a notation borrowed from probability theory to indicate sets. For example, if $f, g : [a,b] \to \mathbb{R}$, then $\{f \le 1\} = \{x \in [a,b] : f(x) \le 1\}$, $\mu\{f > 0,\ g < 0\}$ is the measure of the set $\{x \in [a,b] : f(x) > 0,\ g(x) < 0\}$, etc.

Given a measurable function $f : [a,b] \to \mathbb{C}$, the essential range of f is denoted by $\mathcal{ER}(f)$. We recall that $\mathcal{ER}(f)$ is defined as

$$\mathcal{ER}(f) = \{z \in \mathbb{C} : \mu\{|f - z| < \varepsilon\} > 0 \text{ for all } \varepsilon > 0\}.$$

It is clear that $\mathcal{ER}(f) \subseteq \overline{f([a,b])}$. Moreover, $\mathcal{ER}(f)$ is closed and $f \in \mathcal{ER}(f)$ a.e.; see, e.g., [13, Lemma 2.1]. If f is real a.e. then $\mathcal{ER}(f)$ is a subset of $\mathbb{R}$. In this case, we define the essential infimum (resp., supremum) of f on $[a,b]$ as the infimum (resp., supremum) of $\mathcal{ER}(f)$:

$$\operatorname*{ess\,inf}_{[a,b]} f = \inf \mathcal{ER}(f), \qquad \operatorname*{ess\,sup}_{[a,b]} f = \sup \mathcal{ER}(f).$$

Throughout this paper, any finite sequence of points in $\mathbb{R}$ is referred to as a grid. Consider an interval $[a,b]$ and, for every n, let $\mathcal{G}_n = \{x_{i,n}\}_{i=1,\dots,d_n}$ be a grid of d_n points in $[a,b]$ with $d_n \to \infty$ as $n \to \infty$. The number

$$m(\mathcal{G}_n) = \max_{i=1,\dots,d_n} \left| x_{i,n} - \left(a + i\,\frac{b-a}{d_n+1} \right) \right|$$

measures the distance of $\mathcal{G}_n$ from the uniform grid $\{a + i(b-a)/(d_n+1)\}_{i=1,\dots,d_n}$; we refer to it as the uniformity measure of the grid $\mathcal{G}_n$. We say that $\mathcal{G}_n$ is asymptotically uniform (a.u.) in $[a,b]$ if

$$\lim_{n\to\infty} m(\mathcal{G}_n) = 0.$$

2.2. Asymptotic singular value and eigenvalue distributions of a matrix-sequence

Throughout this paper, a matrix-sequence is a sequence of the form $\{A_n\}_n$, where A_n is a square matrix and $\text{size}(A_n) = d_n \to \infty$ as $n \to \infty$. We denote by $C_c(\mathbb{R})$ (resp., $C_c(\mathbb{C})$) the space of continuous complex-valued functions with bounded support defined on $\mathbb{R}$ (resp., $\mathbb{C}$). If $A \in \mathbb{C}^{m\times m}$, the singular values and eigenvalues of A are denoted by $\sigma_1(A), \dots, \sigma_m(A)$ and $\lambda_1(A), \dots, \lambda_m(A)$, respectively. The minimum and maximum singular values of A are also denoted by $\sigma_{\min}(A)$ and $\sigma_{\max}(A)$. A matrix-valued function $f : [a,b] \to \mathbb{C}^{k\times k}$ is said to be measurable (resp., bounded, continuous, continuous a.e., in $L^p([a,b])$, etc.) if its components $f_{ij} : [a,b] \to \mathbb{C}$, $i,j = 1,\dots,k$, are measurable (resp., bounded, continuous, continuous a.e., in $L^p([a,b])$, etc.).

Definition 2.1 *(asymptotic singular value and eigenvalue distributions of a matrix-sequence).* Let $\{A_n\}_n$ be a matrix-sequence with A_n of size d_n, and let $f : [a,b] \to \mathbb{C}^{k\times k}$ be measurable.

- We say that $\{A_n\}_n$ has an asymptotic eigenvalue (or spectral) distribution described by f if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\lambda_i(A_n)) = \frac{1}{b-a} \int_a^b \frac{\sum_{i=1}^k F(\lambda_i(f(x)))}{k}\,\mathrm{d}x, \qquad \forall\, F \in C_c(\mathbb{C}). \tag{2.1}$$

  In this case, f is called the eigenvalue (or spectral) symbol of $\{A_n\}_n$ and we write $\{A_n\}_n \sim_\lambda f$.
- We say that $\{A_n\}_n$ has an asymptotic singular value distribution described by f if

$$\lim_{n\to\infty} \frac{1}{d_n} \sum_{i=1}^{d_n} F(\sigma_i(A_n)) = \frac{1}{b-a} \int_a^b \frac{\sum_{i=1}^k F(\sigma_i(f(x)))}{k}\,\mathrm{d}x, \qquad \forall\, F \in C_c(\mathbb{R}). \tag{2.2}$$

  In this case, f is called the singular value symbol of $\{A_n\}_n$ and we write $\{A_n\}_n \sim_\sigma f$.

We remark that Definition 2.1 is well-posed as the functions $x \mapsto \sum_{i=1}^k F(\lambda_i(f(x)))$ and $x \mapsto \sum_{i=1}^k F(\sigma_i(f(x)))$ appearing in (2.1)–(2.2) are measurable [2, Lemma 2.1]. Throughout this paper, whenever we write a relation such as $\{A_n\}_n \sim_\lambda f$ or $\{A_n\}_n \sim_\sigma f$, it is understood that $\{A_n\}_n$ and f are as in Definition 2.1, i.e., $\{A_n\}_n$ is a matrix-sequence and f is a measurable function taking values in $\mathbb{C}^{k\times k}$ for some k and defined on some compact real interval $[a,b]$.

Remark 2.1. The informal meaning behind the asymptotic spectral distribution (2.1) is the following: assuming that f possesses k a.e. continuous eigenvalue functions $\lambda_i(f(x))$, $i = 1, \ldots, k$, the eigenvalues of A_n, except possibly for $o(d_n)$ outliers, can be subdivided into k different subsets of approximately the same cardinality; and, for n large enough, the eigenvalues belonging to the ith subset are approximately equal to the samples of the ith eigenvalue function $\lambda_i(f(x))$ over a uniform grid in the domain $[a, b]$. For instance, if $d_n = nk$ then, assuming we have no outliers, the eigenvalues of A_n are approximately equal to

$$\lambda_i\Big(f\Big(a + j\,\frac{b-a}{n}\Big)\Big), \qquad j = 1, \ldots, n, \qquad i = 1, \ldots, k.$$

A completely analogous meaning can also be given for the asymptotic singular value distribution (2.2). A noteworthy class of matrix-sequences $\{A_n\}_n$ that, under suitable assumptions, enjoy an asymptotic singular value and spectral distribution described by a function f taking values in $\mathbb{C}^{k\times k}$ is the class of $k \times k$ block generalized locally Toeplitz (GLT) sequences [2].

Since any finite multiset of numbers can always be interpreted as the spectrum of a matrix, a byproduct of Definition 2.1 is the following definition.

Definition 2.2 *(asymptotic distribution of a sequence of finite multisets of numbers).* Let $\{\Lambda_n = \{\lambda_{1,n}, \ldots, \lambda_{d_n,n}\}\}_n$ be a sequence of finite multisets of numbers such that $d_n \to \infty$ as $n \to \infty$, and let f be as in Definition 2.1. We say that $\{\Lambda_n\}_n$ has an asymptotic distribution described by f, and we write $\{\Lambda_n\}_n \sim f$, if $\{A_n\}_n \sim_\lambda f$, where A_n is any matrix whose spectrum equals Λ_n (e.g., $A_n = \mathrm{diag}(\lambda_{1,n}, \ldots, \lambda_{d_n,n})$).

The next lemma is a slight generalization of [1, Lemma 3.12] and it can be proved in the same way.

Lemma 2.1. *Let $\{A_n\}_n$ be a matrix-sequence, let $f : [a, b] \to \mathbb{C}^{k\times k}$ be measurable, and suppose that $\{A_n\}_n \sim_\lambda f$. Let $\lambda_1(f), \ldots, \lambda_k(f) : [a, b] \to \mathbb{C}$ be k measurable functions such that $\lambda_1(f(x)), \ldots, \lambda_k(f(x))$ are the eigenvalues of $f(x)$ for every $x \in [a, b]$. Then, for every $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, we have $\{A_n\}_n \sim_\lambda \tilde{f}$, where $\tilde{f}$ is the following concatenation on the interval $[\alpha, \beta]$ of resized versions of $\lambda_1(f), \ldots, \lambda_k(f)$:*

$$\tilde{f} : [\alpha, \beta] \to \mathbb{C}, \qquad \tilde{f}(y) = \begin{cases} \lambda_1(f(a + \frac{(b-a)k}{\beta-\alpha}(y-\alpha))), & \alpha \le y < \alpha + \frac{\beta-\alpha}{k}, \\ \lambda_2(f(a + \frac{(b-a)k}{\beta-\alpha}(y-\alpha-\frac{\beta-\alpha}{k}))), & \alpha + \frac{\beta-\alpha}{k} \le y < \alpha + 2\frac{\beta-\alpha}{k}, \\ \lambda_3(f(a + \frac{(b-a)k}{\beta-\alpha}(y-\alpha-2\frac{\beta-\alpha}{k}))), & \alpha + 2\frac{\beta-\alpha}{k} \le y < \alpha + 3\frac{\beta-\alpha}{k}, \\ \quad\vdots & \quad\vdots \\ \lambda_k(f(a + \frac{(b-a)k}{\beta-\alpha}(y-\alpha-(k-1)\frac{\beta-\alpha}{k}))), & \alpha + (k-1)\frac{\beta-\alpha}{k} \le y \le \beta. \end{cases}$$

Theorems 2.1–2.2 are fundamental asymptotic distribution results obtained in [1]. They play a central role hereinafter. Throughout this paper, we use “increasing” as a synonym of “non-decreasing”.

Theorem 2.1. *Let $f : [a, b] \to \mathbb{R}$ be bounded and continuous a.e. with $\mathcal{ER}(f) = [\inf_{[a,b]} f, \sup_{[a,b]} f]$. Let $\{\Lambda_n = \{\lambda_{1,n}, \ldots, \lambda_{d_n,n}\}\}_n$ be a sequence of finite multisets of real numbers such that $d_n \to \infty$ as $n \to \infty$. Assume the following.*

- $\{\Lambda_n\}_n \sim f$.
- $\Lambda_n \subseteq [\inf_{[a,b]} f - \varepsilon_n, \sup_{[a,b]} f + \varepsilon_n]$ *for every n and for some $\varepsilon_n \to 0$ as $n \to \infty$.*

Then, for every a.u. grid $\{x_{i,n}\}_{i=1,\ldots,d_n}$ in $[a, b]$, if σ_n and τ_n are two permutations of $\{1, \ldots, d_n\}$ such that the vectors $[f(x_{\sigma_n(1),n}), \ldots, f(x_{\sigma_n(d_n),n})]$ and $[\lambda_{\tau_n(1),n}, \ldots, \lambda_{\tau_n(d_n),n}]$ are sorted in increasing order, we have

$$\max_{i=1,\ldots,d_n} |f(x_{\sigma_n(i),n}) - \lambda_{\tau_n(i),n}| \to 0 \ \text{ as } \ n \to \infty.$$

In particular,

$$\min_{\tau} \max_{i=1,\ldots,d_n} |f(x_{i,n}) - \lambda_{\tau(i),n}| \to 0 \ \text{ as } \ n \to \infty,$$

where the minimum is taken over all permutations τ of $\{1, \ldots, d_n\}$.

To properly state Theorem 2.2, we need the following definition.

Definition 2.3 *(local extremum points).* Given a function $f : [a, b] \to \mathbb{R}$ and a point $x_0 \in [a, b]$, we say that x_0 is a local maximum point (resp., local minimum point) for f if $f(x_0) \ge f(x)$ (resp., $f(x_0) \le f(x)$) for all x belonging to a neighborhood of x_0 in $[a, b]$.

We point out that, according to Definition 2.3, by “local maximum/minimum point” we mean “weak local maximum/minimum point”. For example, if f is constant on $[a,b]$, then all points of $[a,b]$ are both local maximum and local minimum points for f.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ be bounded with a finite number of local maximum points, local minimum points, and discontinuity points, and with $\mathcal{ER}(f) = [\inf_{[a,b]} f, \sup_{[a,b]} f]$. Let $\{\Lambda_n = \{\lambda_{1,n}, \dots, \lambda_{d_n,n}\}\}_n$ be a sequence of finite multisets of real numbers such that $d_n \to \infty$ as $n \to \infty$. Assume the following.*

- $\{\Lambda_n\}_n \sim f$.
- $\Lambda_n \subseteq f([a,b])$ *for every* n.

Then, for every n, there exist an a.u. grid $\{x_{i,n}\}_{i=1,\dots,d_n}$ in $[a,b]$ and a permutation τ_n of $\{1,\dots,d_n\}$ such that

$$\lambda_{\tau_n(i),n} = f(x_{i,n}), \qquad i = 1,\dots,d_n.$$

2.3. Toeplitz matrices

It is not difficult to see that the operator $T_n(\cdot) : L^1([-\pi,\pi]) \to \mathbb{C}^{n\times n}$, which associates with each $f \in L^1([-\pi,\pi])$ the corresponding $n\times n$ Toeplitz matrix $T_n(f)$, is linear and satisfies $T_n(1) = I_n$, where I_n is the $n\times n$ identity matrix. Moreover, the conjugate transpose of $T_n(f)$ is given by

$$T_n(f)^* = T_n(\overline{f})$$

for every $f \in L^1([-\pi,\pi])$ and every n; see, e.g., [13, Section 6.2]. In particular, if f is real a.e., then $\overline{f} = f$ a.e. and the matrices $T_n(f)$ are Hermitian. Moreover, if f is real a.e. and even, then its Fourier coefficients are real and even (see Lemma 3.4 below), and therefore the matrices $T_n(f)$ are real and symmetric. The next theorems collect relevant properties of Toeplitz matrices generated by a function. For the related proofs, see [13, Theorems 6.1 and 6.5].

Theorem 2.3. *For every $f \in L^1([-\pi,\pi])$ we have $\{T_n(f)\}_n \sim_\sigma f$.*

Theorem 2.4. *Let $f \in L^1([-\pi,\pi])$ be real and let*

$$m_f = \operatorname*{ess\,inf}_{[-\pi,\pi]} f, \qquad M_f = \operatorname*{ess\,sup}_{[-\pi,\pi]} f.$$

Then, the following properties hold.

1. *$T_n(f)$ is Hermitian and the eigenvalues of $T_n(f)$ lie in the interval $[m_f, M_f]$ for all n.*
2. *If f is not a.e. constant, then $m_f < M_f$ and the eigenvalues of $T_n(f)$ lie in (m_f, M_f) for all n.*
3. $\{T_n(f)\}_n \sim_\lambda f$.

The next localization results for the singular values of Toeplitz matrices have been proved in [31, Lemma I.2] and [29, Theorem 4.4], respectively. Throughout this paper, if S is any subset of $\mathbb{C}$, we denote by $\mathrm{Co}(S)$ its convex hull.

Lemma 2.2. *Let $f \in L^1([-\pi,\pi])$ and let d_f be the distance of 0 from $\mathrm{Co}(\mathcal{ER}(f))$ in $\mathbb{C}$. Then, the singular values of $T_n(f)$ lie in $[d_f, M_{|f|}]$ for all n, where $M_{|f|} = \operatorname{ess\,sup}_{[-\pi,\pi]} |f|$.*

Lemma 2.3. *Let $f : [-\pi,\pi] \to \mathbb{C}$ belong to the Krein algebra, i.e., $f \in L^\infty([-\pi,\pi])$ and the Fourier coefficients of f satisfy*

$$\sum_{k\in\mathbb{Z}} |k|\,|f_k|^2 < \infty.$$

Then, for every $\varepsilon > 0$ there exists a constant C_ε such that the singular values of $T_n(f)$ lie in the ε-expansion of $\mathcal{ER}(|f|)$ given by

$$(\mathcal{ER}(|f|))_\varepsilon = \{s \in [0,\infty) : |s-t| < \varepsilon \text{ for some } t \in \mathcal{ER}(|f|)\}$$

except for at most C_ε outliers.

2.4. Flipped Toeplitz matrices

If $f : [-\pi,\pi] \to \mathbb{C}$ is a function in $L^1([-\pi,\pi])$ and $n \in \mathbb{N}$, we define the Hankel matrix

$$H_n(f) = Y_n T_n(f),$$

where $Y_n = H_n(1)$ is the $n\times n$ exchange matrix in (1.2). We remark that our definition of $H_n(f)$ is different from the standard definition of Hankel matrices generated by a function f [4, Section 11.4]. We therefore refer to $H_n(f)$ as the flipped Toeplitz matrix generated by f rather than the Hankel matrix generated by f. A vector $\boldsymbol{v}\in\mathbb{R}^n$ is called symmetric (resp., skew-symmetric) if $Y_n\boldsymbol{v}=\boldsymbol{v}$ (resp., $Y_n\boldsymbol{v}=-\boldsymbol{v}$).

Remark 2.2 *(eigendecomposition of the exchange matrix).* Let $\mathbb{V}_n^+$ and $\mathbb{V}_n^-$ be the subspaces of $\mathbb{R}^n$ consisting of symmetric and skew-symmetric vectors, respectively:

$$\mathbb{V}_n^+ = \{\boldsymbol{v}\in\mathbb{R}^n : Y_n\boldsymbol{v}=\boldsymbol{v}\},\qquad \mathbb{V}_n^- = \{\boldsymbol{v}\in\mathbb{R}^n : Y_n\boldsymbol{v}=-\boldsymbol{v}\}.$$

It is easy to see that $\dim\mathbb{V}_n^+=\lceil n/2\rceil$ and $\dim\mathbb{V}_n^-=\lfloor n/2\rfloor$. Indeed, a basis for $\mathbb{V}_n^+$ is

$$\boldsymbol{e}_i+\boldsymbol{e}_{n-i+1},\qquad i=1,\dots,\lceil n/2\rceil,$$

and a basis for $\mathbb{V}_n^-$ is

$$\boldsymbol{e}_i-\boldsymbol{e}_{n-i+1},\qquad i=1,\dots,\lfloor n/2\rfloor,$$

where $\boldsymbol{e}_1,\dots,\boldsymbol{e}_n$ are the vectors of the canonical basis of $\mathbb{R}^n$. Note that $\mathbb{V}_n^+$ and $\mathbb{V}_n^-$ are, by definition, eigenspaces of Y_n associated with the eigenvalues 1 and -1, respectively, and we have $\dim\mathbb{V}_n^+ + \dim\mathbb{V}_n^- = n$. This yields the eigendecomposition of the exchange matrix Y_n, which has only two distinct eigenvalues 1 and -1 with corresponding eigenspaces $\mathbb{V}_n^+$ and $\mathbb{V}_n^-$. We can thus write the eigendecomposition of Y_n as follows:

$$Y_n = V_n\Delta_n V_n^{-1},\qquad \Delta_n = \operatorname*{diag}_{i=1,\dots,n}(-1)^{i+1},\qquad V_n=\left[\boldsymbol{v}_1\,|\,\boldsymbol{v}_2\,|\,\cdots\,|\,\boldsymbol{v}_n\right],\tag{2.3}$$

where $\{\boldsymbol{v}_i : i \text{ is odd}\}$ is any basis of $\mathbb{V}_n^+$ and $\{\boldsymbol{v}_i : i \text{ is even}\}$ is any basis of $\mathbb{V}_n^-$.

If $f:[-\pi,\pi]\to\mathbb{C}$ is a function in $L^1([-\pi,\pi])$ with real Fourier coefficients, then $T_n(f)$ is real for all n. In this case, $H_n(f)$ is real and symmetric for every n, and the next theorem appeared in [9,22] gives the asymptotic spectral distribution of the matrix-sequence $\{H_n(f)\}_n$.

Theorem 2.5. *If f is a function in $L^1([-\pi,\pi])$ with real Fourier coefficients then $\{H_n(f)\}_n\sim_\lambda H$, where*

$$H:[-\pi,\pi]\to\mathbb{C}^{2\times 2},\qquad H(x)=\operatorname{diag}(|f(x)|,-|f(x)|).$$

The next lemma is a corollary of the Cantoni–Butler theorem [5, Theorem 2]; see also [33, Section 5]. An $n\times n$ matrix A is called centrosymmetric if it is symmetric with respect to its center, i.e., $A_{ij}=A_{n-i+1,n-j+1}$ for all $i,j=1,\dots,n$. Equivalently, A is centrosymmetric if $Y_nAY_n=A$. Note that any symmetric Toeplitz matrix is centrosymmetric.

Lemma 2.4. *Let T_n be a real symmetric Toeplitz matrix of size n and let $H_n=Y_nT_n$. Then, the following properties hold.*

1. *There exists an orthonormal basis of $\mathbb{R}^n$ consisting of eigenvectors of T_n such that $\lceil n/2\rceil$ vectors of this basis are symmetric and the other $\lfloor n/2\rfloor$ vectors are skew-symmetric.*
2. *Let $\{\boldsymbol{v}_1,\dots,\boldsymbol{v}_n\}$ be a basis of $\mathbb{R}^n$ such that, setting $V_n=\left[\boldsymbol{v}_1\,|\,\boldsymbol{v}_2\,|\,\cdots\,|\,\boldsymbol{v}_n\right]$, we have:*
   - *$\boldsymbol{v}_i$ is alternately symmetric or skew-symmetric (starting with symmetric), i.e., in view of Remark 2.2,*

     $$Y_n=V_n\Delta_nV_n^*,\qquad \Delta_n=\operatorname*{diag}_{i=1,\dots,n}(-1)^{i+1};$$

   - *$T_n\boldsymbol{v}_i=\lambda_i(T_n)\boldsymbol{v}_i$ for $i=1,\dots,n$, i.e.,*

     $$T_n=V_nD_nV_n^*,\qquad D_n=\operatorname*{diag}_{i=1,\dots,n}\lambda_i(T_n).$$

   Then,
   - *$H_n\boldsymbol{v}_i=\lambda_i(H_n)\boldsymbol{v}_i$ with $\lambda_i(H_n)=(-1)^{i+1}\lambda_i(T_n)$ for $i=1,\dots,n$, i.e.,*

     $$H_n=V_nE_nV_n^*,\qquad E_n=\Delta_nD_n.$$

Proof. Since T_n is symmetric centrosymmetric, the first property follows from [5, Theorem 2]. The second property is a consequence of the first property and the definition $H_n=Y_nT_n$. □

Taking into account that the matrices $T_n(f)$ are real and symmetric whenever f is real and even, the following result is a corollary of Lemma 2.4 and Theorem 2.4.

Corollary 2.1. *Let $f \in L^1([-\pi,\pi])$ be real and even. Then, for every n, there exists a real unitary matrix V_n such that*

$$Y_n = V_n \Delta_n V_n^*, \qquad \Delta_n = \operatorname*{diag}_{i=1,\dots,n} (-1)^{i+1}, \tag{2.4}$$

$$T_n(f) = V_n D_n V_n^*, \qquad D_n = \operatorname*{diag}_{i=1,\dots,n} \lambda_i(T_n(f)), \tag{2.5}$$

$$H_n(f) = V_n E_n V_n^*, \qquad E_n = \Delta_n D_n = \operatorname*{diag}_{i=1,\dots,n} \lambda_i(H_n(f)), \tag{2.6}$$

$$\lambda_i(H_n(f)) = (-1)^{i+1}\lambda_i(T_n(f)), \qquad i=1,\dots,n. \tag{2.7}$$

In particular, if f is not a.e. constant, the eigenvalues of $H_n(f)$ lie in $(-M_f,-m_f)\cup(m_f,M_f)$, where

$$m_f = \operatorname*{ess\,inf}_{[0,\pi]} f, \qquad M_f = \operatorname*{ess\,sup}_{[0,\pi]} f.$$

3. Main results: spectral properties of flipped Toeplitz matrices

In this section, we state and prove the main results of this paper. Theorem 3.1 is our first main result. It shows, among others, that, for a real even function f bounded from below or above, the eigenvalues of $H_n(f)$ can be subdivided into two subsets of cardinalities $\lceil n/2\rceil$ and $\lfloor n/2\rfloor$ such that

- the $\lceil n/2\rceil$ eigenvalues belonging to the first subset coincide with $\lceil n/2\rceil$ eigenvalues of $T_n(f)$ and have an asymptotic distribution described by f;
- the $\lfloor n/2\rfloor$ eigenvalues belonging to the second subset coincide with $\lfloor n/2\rfloor$ eigenvalues of $-T_n(f)$ and have an asymptotic distribution described by $-f$.

Throughout this paper, if Λ is a multiset of numbers and $\alpha\in\mathbb{C}$, we denote by $\Lambda+\alpha$ and $\Lambda-\alpha$ the multisets $\{\lambda+\alpha : \lambda\in\Lambda\}$ and $\{\lambda-\alpha : \lambda\in\Lambda\}$, respectively.

Theorem 3.1. *Let $f \in L^1([-\pi,\pi])$ be real and even, and suppose that $m_f = \operatorname{ess\,inf}_{[0,\pi]} f > -\infty$ or $M_f = \operatorname{ess\,sup}_{[0,\pi]} f < \infty$. Then, for every n, there exists a real unitary matrix V_n such that (2.4)–(2.7) hold and*

$$\{\Lambda_n^+\}_n \sim f, \qquad \Lambda_n^+ = \{\lambda_{2i-1}(H_n(f))\}_{i=1,\dots,\lceil n/2\rceil} = \{\lambda_{2i-1}(T_n(f))\}_{i=1,\dots,\lceil n/2\rceil}, \tag{3.1}$$

$$\{\Lambda_n^-\}_n \sim -f, \qquad \Lambda_n^- = \{\lambda_{2i}(H_n(f))\}_{i=1,\dots,\lfloor n/2\rfloor} = \{-\lambda_{2i}(T_n(f))\}_{i=1,\dots,\lfloor n/2\rfloor}. \tag{3.2}$$

Proof. We prove the theorem in the case where $m_f > -\infty$ (the proof in the case where $M_f < \infty$ is similar). Let $g = f+\alpha$, where $\alpha\in\mathbb{R}$ is a constant such that $m_f+\alpha>0$. Note that g is a real even function in $L^1([-\pi,\pi])$ like f, and moreover $g\ge m_f+\alpha>0$ a.e. in $[0,\pi]$, so $\mathcal{ER}(g)$ is a closed subset of $(0,\infty)$. By Corollary 2.1, for every n, there exists a real unitary matrix V_n such that

$$Y_n = V_n \Delta_n V_n^*, \qquad \Delta_n = \operatorname*{diag}_{i=1,\dots,n} (-1)^{i+1}, \tag{3.3}$$

$$T_n(g) = V_n D_n' V_n^*, \qquad D_n' = \operatorname*{diag}_{i=1,\dots,n} \lambda_i(T_n(g)), \tag{3.4}$$

$$H_n(g) = V_n E_n' V_n^*, \qquad E_n' = \Delta_n D_n' = \operatorname*{diag}_{i=1,\dots,n} \lambda_i(H_n(g)), \tag{3.5}$$

$$\lambda_i(H_n(g)) = (-1)^{i+1}\lambda_i(T_n(g)), \qquad i=1,\dots,n. \tag{3.6}$$

Let

$$\tilde\Lambda_n^+ = \{\lambda_{2i-1}(H_n(g))\}_{i=1,\dots,\lceil n/2\rceil} = \{\lambda_{2i-1}(T_n(g))\}_{i=1,\dots,\lceil n/2\rceil},$$
$$\tilde\Lambda_n^- = \{\lambda_{2i}(H_n(g))\}_{i=1,\dots,\lfloor n/2\rfloor} = \{-\lambda_{2i}(T_n(g))\}_{i=1,\dots,\lfloor n/2\rfloor}.$$

We prove that $\{\tilde\Lambda_n^+\}_n\sim g$. For every $F\in C_c(\mathbb{C})$, let $\tilde F\in C_c(\mathbb{C})$ be a function such that $\tilde F = F$ on $[m_f+\alpha,\infty)$ and $\tilde F = 0$ on $(-\infty,-m_f-\alpha]$. Note that $\tilde\Lambda_n^+\subseteq[m_f+\alpha,\infty)$ and $\tilde\Lambda_n^-\subseteq(-\infty,-m_f-\alpha]$ by Theorem 2.4. By Theorem 2.5 and the fact that g is a real even function,

$$\lim_{n\to\infty}\frac{1}{\lceil n/2\rceil}\sum_{i=1}^{\lceil n/2\rceil}F(\lambda_{2i-1}(H_n(g))) = \lim_{n\to\infty}\frac{1}{\lceil n/2\rceil}\sum_{i=1}^{\lceil n/2\rceil}\tilde F(\lambda_{2i-1}(H_n(g))) = \lim_{n\to\infty}\frac{n}{\lceil n/2\rceil}\cdot\frac1n\sum_{i=1}^{n}\tilde F(\lambda_i(H_n(g)))$$
$$= 2\cdot\frac1\pi\int_0^\pi\frac{\tilde F(g(x))+\tilde F(-g(x))}{2}\mathrm{d}x = \frac1\pi\int_0^\pi\tilde F(g(x))\mathrm{d}x = \frac1\pi\int_0^\pi F(g(x))\mathrm{d}x.$$

Hence, $\{\tilde\Lambda_n^+\}_n\sim g$. Similarly, one can show that $\{\tilde\Lambda_n^-\}_n\sim -g$.

To conclude the proof, we note that, by (3.3)–(3.4) and the linearity of the operator $T_n(\cdot)$,

$$T_n(f) = T_n(g-\alpha) = T_n(g) - \alpha I_n = V_n(D_n' - \alpha I_n)V_n^*, \tag{3.7}$$

$$H_n(f) = Y_n T_n(f) = V_n \Delta_n (D_n' - \alpha I_n)V_n^*. \tag{3.8}$$

Define the following ordering for the eigenvalues of $T_n(f)$ and $H_n(f)$:

$$\lambda_i(T_n(f)) = \lambda_i(T_n(g)) - \alpha, \qquad i = 1,\dots,n, \tag{3.9}$$

$$\lambda_i(H_n(f)) = (-1)^{i+1}(\lambda_i(T_n(g)) - \alpha) = (-1)^{i+1}\lambda_i(T_n(f)), \qquad i = 1,\dots,n. \tag{3.10}$$

In view of (3.7)–(3.8), it is now easy to check that (2.4)–(2.7) are satisfied with

$$D_n = D_n' - \alpha I_n,$$

$$E_n = \Delta_n(D_n' - \alpha I_n) = \Delta_n D_n.$$

Moreover, also (3.1)–(3.2) are satisfied, because $\{\tilde\Lambda_n^+\}_n \sim g$ is equivalent to $\{\Lambda_n^+\}_n \sim f$ and $\{\tilde\Lambda_n^-\}_n \sim -g$ is equivalent to $\{\Lambda_n^-\}_n \sim -f$. These equivalences follow from Definition 2.2, the equation $g = f + \alpha$, and the observation that

$$\tilde\Lambda_n^+ = \{\lambda_{2i-1}(T_n(g))\}_{i=1,\dots,\lceil n/2\rceil} = \{\lambda_{2i-1}(T_n(f)) + \alpha\}_{i=1,\dots,\lceil n/2\rceil} = \Lambda_n^+ + \alpha,$$

$$\tilde\Lambda_n^- = \{-\lambda_{2i}(T_n(g))\}_{i=1,\dots,\lceil n/2\rceil} = \{-\lambda_{2i}(T_n(f)) - \alpha\}_{i=1,\dots,\lceil n/2\rceil} = \Lambda_n^- - \alpha. \quad \square$$

Theorem 3.2 is our second main result. It shows, among others, that, for a real even bounded and a.e. continuous function f, the eigenvalues of $H_n(f)$ are given by the following alternating sign relationship:

$$\lambda_i(H_n(f)) = (-1)^{i+1}\lambda_i(T_n(f)) = (-1)^{i+1} f(x_{i,n}) + \varepsilon_{i,n}, \qquad i = 1,\dots,n, \tag{3.11}$$

where $\{x_{i,n}\}_{i=1,\dots,n}$ is any a.u. grid and $\max_{i=1,\dots,n}|\varepsilon_{i,n}| \to 0$ as $n \to \infty$. To prove Theorem 3.2, we need the following lemma about a.u. grids. Throughout this paper, if $\{a_n\}_n$ and $\{b_n\}_n$ are any two numerical sequences such that $a_n, b_n \neq 0$ eventually as $n \to \infty$, we write $a_n \sim b_n$ as $n \to \infty$ to indicate that $a_n/b_n \to 1$ as $n \to \infty$.

Lemma 3.1. *Let $\{x_{i,n}\}_{i=1,\dots,d_n}$ and $\{y_{i,n}\}_{i=1,\dots,e_n}$ be two grids, with $d_n \to \infty$ and $e_n \sim d_n$ as $n \to \infty$. Let $\{z_{i,n}\}_{i=1,\dots,d_n+e_n}$ be the grid obtained by alternately picking one point from the first grid and one point from the second grid, and by positioning the remaining points of the largest grid at the end. In formulas,*

$$\{z_{i,n}\}_{i=1,\dots,d_n+e_n} = \begin{cases} \{x_{1,n}, y_{1,n}, x_{2,n}, y_{2,n}, \dots, x_{d_n,n}, y_{d_n,n}, y_{d_n+1,n}, \dots, y_{e_n,n}\}, & \text{if } d_n \le e_n, \\ \{x_{1,n}, y_{1,n}, x_{2,n}, y_{2,n}, \dots, x_{e_n,n}, y_{e_n,n}, x_{e_n+1,n}, \dots, x_{d_n,n}\}, & \text{if } d_n > e_n. \end{cases}$$

Then, for every real interval $[a,b]$, the grid $\{z_{i,n}\}_{i,\dots,d_n+e_n}$ is a.u. in $[a,b]$ if and only if the grids $\{x_{i,n}\}_{i=1,\dots,d_n}$ and $\{y_{i,n}\}_{i=1,\dots,e_n}$ are a.u. in $[a,b]$.

Proof. Set

$$\mathcal{X}_n = \{x_{i,n}\}_{i=1,\dots,d_n}, \qquad \mathcal{Y}_n = \{y_{i,n}\}_{i=1,\dots,e_n}, \qquad \mathcal{Z}_n = \{z_{i,n}\}_{i=1,\dots,d_n+e_n},$$

$$\xi_{i,n} = a + i\,\frac{b-a}{d_n+1}, \qquad \psi_{i,n} = a + i\,\frac{b-a}{e_n+1}, \qquad \zeta_{i,n} = a + i\,\frac{b-a}{d_n+e_n+1}.$$

Note that the uniformity measures of the grids $\mathcal{X}_n$, $\mathcal{Y}_n$, $\mathcal{Z}_n$ with respect to the interval $[a,b]$ are given by

$$m(\mathcal{X}_n) = \max_{i=1,\dots,d_n}|x_{i,n} - \xi_{i,n}|, \qquad m(\mathcal{Y}_n) = \max_{i=1,\dots,e_n}|y_{i,n} - \psi_{i,n}|, \qquad m(\mathcal{Z}_n) = \max_{i=1,\dots,d_n+e_n}|z_{i,n} - \zeta_{i,n}|.$$

Before proving the lemma, we make the following observations.

- For $i = 1,\dots,\min(d_n,e_n)$,

$$\begin{aligned} |\xi_{i,n} - \zeta_{2i-1,n}| &= (b-a)\left|\frac{i}{d_n+1} - \frac{2i-1}{d_n+e_n+1}\right| = (b-a)\left|\frac{i(e_n-d_n)+d_n-i+1}{(d_n+1)(d_n+e_n+1)}\right| \\ &\le (b-a)\frac{d_n|e_n-d_n|+d_n}{(d_n+1)(d_n+e_n+1)} = \frac{(b-a)d_n^2}{(d_n+1)(d_n+e_n+1)}\left(\left|\frac{e_n}{d_n}-1\right| + \frac{1}{d_n}\right) = \alpha_n, \end{aligned}$$

  where α_n depends only on n and $\alpha_n \to 0$ as $n \to \infty$ due to the assumptions that $d_n \to \infty$ and $e_n \sim d_n$ as $n \to \infty$. Similarly, for $i = 1,\dots,\min(d_n,e_n)$,

$$|\psi_{i,n}-\zeta_{2i,n}| = (b-a)\left|\frac{i}{e_n+1}-\frac{2i}{d_n+e_n+1}\right| = (b-a)\left|\frac{i(d_n-e_n)-i}{(e_n+1)(d_n+e_n+1)}\right|$$
$$\le (b-a)\frac{e_n|d_n-e_n|+e_n}{(e_n+1)(d_n+e_n+1)} = \frac{(b-a)e_n^2}{(e_n+1)(d_n+e_n+1)}\left(\left|\frac{d_n}{e_n}-1\right|+\frac{1}{e_n}\right) = \beta_n,$$

where β_n depends only on n and $\beta_n \to 0$ as $n\to\infty$ due to the assumptions that $d_n\to\infty$ and $e_n \sim d_n$ as $n\to\infty$.

- If $d_n \le e_n$ then, for every $i = d_n+1,\dots,e_n$,

$$|\psi_{i,n}-\zeta_{2d_n+(i-d_n),n}| = (b-a)\left|\frac{i}{e_n+1}-\frac{d_n+i}{d_n+e_n+1}\right| = (b-a)\left|\frac{d_n(i-e_n)-d_n}{(e_n+1)(d_n+e_n+1)}\right|$$
$$\le (b-a)\frac{d_n(e_n-d_n)+d_n}{(e_n+1)(d_n+e_n+1)} = \frac{(b-a)d_n^2}{(e_n+1)(d_n+e_n+1)}\left(\frac{e_n}{d_n}-1+\frac{1}{d_n}\right) = \gamma_n,$$

where γ_n depends only on n and $\gamma_n\to 0$ as $n\to\infty$ due to the assumptions that $d_n\to\infty$ and $e_n\sim d_n$ as $n\to\infty$. Similarly, if $d_n > e_n$ then, for every $i = e_n+1,\dots,d_n$,

$$|\xi_{i,n}-\zeta_{2e_n+(i-e_n),n}| = (b-a)\left|\frac{i}{d_n+1}-\frac{e_n+i}{d_n+e_n+1}\right| = (b-a)\left|\frac{e_n(i-d_n)-e_n}{(d_n+1)(d_n+e_n+1)}\right|$$
$$\le (b-a)\frac{e_n(d_n-e_n)+e_n}{(d_n+1)(d_n+e_n+1)} = \frac{(b-a)e_n^2}{(d_n+1)(d_n+e_n+1)}\left(\frac{d_n}{e_n}-1+\frac{1}{e_n}\right) = \delta_n,$$

where δ_n depends only on n and $\delta_n\to 0$ as $n\to\infty$ due to the assumptions that $d_n\to\infty$ and $e_n\sim d_n$ as $n\to\infty$.

We now prove the lemma. For every $i = 1,\dots,\min(d_n,e_n)$, we have

$$\Big||x_{i,n}-\xi_{i,n}|-|z_{2i-1,n}-\zeta_{2i-1,n}|\Big| = \Big||x_{i,n}-\xi_{i,n}|-|x_{i,n}-\zeta_{2i-1,n}|\Big| \le |\xi_{i,n}-\zeta_{2i-1,n}| \le \alpha_n,$$
$$\Big||y_{i,n}-\psi_{i,n}|-|z_{2i,n}-\zeta_{2i,n}|\Big| = \Big||y_{i,n}-\psi_{i,n}|-|y_{i,n}-\zeta_{2i,n}|\Big| \le |\psi_{i,n}-\zeta_{2i,n}| \le \beta_n.$$

Moreover, if $d_n\le e_n$ then, for every $i = d_n+1,\dots,e_n$, we have

$$\Big||y_{i,n}-\psi_{i,n}|-|z_{2d_n+(i-d_n),n}-\zeta_{2d_n+(i-d_n),n}|\Big| = \Big||y_{i,n}-\psi_{i,n}|-|y_{i,n}-\zeta_{2d_n+(i-d_n),n}|\Big|$$
$$\le |\psi_{i,n}-\zeta_{2d_n+(i-d_n),n}| \le \gamma_n.$$

If instead $d_n > e_n$ then, for every $i = e_n+1,\dots,d_n$, we have

$$\Big||x_{i,n}-\xi_{i,n}|-|z_{2e_n+(i-e_n),n}-\zeta_{2e_n+(i-e_n),n}|\Big| = \Big||x_{i,n}-\xi_{i,n}|-|x_{i,n}-\zeta_{2e_n+(i-e_n),n}|\Big|$$
$$\le |\xi_{i,n}-\zeta_{2e_n+(i-e_n),n}| \le \delta_n.$$

In any case, we have

$$\big|\max(m(\mathcal{X}_n), m(\mathcal{Y}_n)) - m(\mathcal{Z}_n)\big| \le \max(\alpha_n,\beta_n,\gamma_n,\delta_n).$$

This implies that $\mathcal{Z}_n$ is a.u. in $[a,b]$ if and only if both $\mathcal{X}_n$ and $\mathcal{Y}_n$ are a.u. in $[a,b]$. □

Theorem 3.2. *Let $f : [-\pi,\pi]\to\mathbb{R}$ be even, bounded and continuous a.e. with $\mathcal{ER}(f) = [\inf_{[0,\pi]} f, \sup_{[0,\pi]} f]$. Then, for every n and every a.u. grid $\{x_{i,n}\}_{i=1,\dots,n}$ in $[0,\pi]$, there exists a real unitary matrix V_n such that (2.4)–(2.7) hold and*

$$\max_{i=1,\dots,n} |f(x_{i,n}) - \lambda_i(T_n(f))| \to 0 \text{ as } n\to\infty. \tag{3.12}$$

Proof. By Theorem 3.1, for every n there exists a real unitary matrix V_n such that (2.4)–(2.7) are satisfied and

$$\{\Lambda_n^+\}_n \sim f, \qquad \Lambda_n^+ = \{\lambda_{2i-1}(H_n(f))\}_{i=1,\dots,\lceil n/2\rceil} = \{\lambda_{2i-1}(T_n(f))\}_{i=1,\dots,\lceil n/2\rceil}, \tag{3.13}$$
$$\{\Lambda_n^-\}_n \sim -f, \qquad \Lambda_n^- = \{\lambda_{2i}(H_n(f))\}_{i=1,\dots,\lfloor n/2\rfloor} = \{-\lambda_{2i}(T_n(f))\}_{i=1,\dots,\lfloor n/2\rfloor}. \tag{3.14}$$

Note that if we permute the columns of V_n and the eigenvalues of $T_n(f)$ and $H_n(f)$ through the same permutation τ of $\{1,\dots,n\}$ such that τ maps odd indices to odd indices and even indices to even indices, then (2.4)–(2.7) continue to hold.

By (3.13)–(3.14), Theorem 2.4, and the assumptions on f, the hypotheses of Theorem 2.1 are satisfied for f and Λ_n^+ as well as for $-f$ and Λ_n^-. Hence, by Theorem 2.1 applied first with f and Λ_n^+ and then with $-f$ and Λ_n^-, we infer that, for every pair of a.u. grids $\{x_{i,n}^+\}_{i=1,\dots,\lceil n/2\rceil}$, $\{x_{i,n}^-\}_{i=1,\dots,\lfloor n/2\rfloor}$ in $[0,\pi]$, we have

$$\min_{\tau}\max_{i=1,\dots,\lceil n/2\rceil} |f(x_{i,n}^+) - \lambda_{2\tau(i)-1}(T_n(f))| \to 0 \text{ as } n\to\infty, \tag{3.15}$$

$$\min_{\tau}\max_{i=1,\dots,\lfloor n/2\rfloor}|-f(x_{i,n}^-)+\lambda_{2\tau(i)}(T_n(f))|\to 0 \text{ as } n\to\infty, \tag{3.16}$$

where the minima are taken over all permutations τ of $\{1,\dots,\lceil n/2\rceil\}$ and $\{1,\dots,\lfloor n/2\rfloor\}$, respectively.

Now let $\{x_{i,n}\}_{i=1,\dots,n}$ be an a.u. grid in $[0,\pi]$. The two subgrids $\{x_{2i-1,n}\}_{i=1,\dots,\lceil n/2\rceil}$, $\{x_{2i,n}\}_{i=1,\dots,\lfloor n/2\rfloor}$ are a.u. in $[0,\pi]$ by Lemma 3.1. We can therefore use these subgrids in (3.15)–(3.16) and we obtain

$$\min_{\tau}\max_{i=1,\dots,\lceil n/2\rceil}|f(x_{2i-1,n})-\lambda_{2\tau(i)-1}(T_n(f))|\to 0 \text{ as } n\to\infty, \tag{3.17}$$

$$\min_{\tau}\max_{i=1,\dots,\lfloor n/2\rfloor}|-f(x_{2i,n})+\lambda_{2\tau(i)}(T_n(f))|\to 0 \text{ as } n\to\infty. \tag{3.18}$$

For every n, we rearrange the eigenvalues of $H_n(f)$ and $T_n(f)$ as follows:

$$\{\lambda_1(H_n(f)),\dots,\lambda_n(H_n(f))\}=\{\lambda_{2\tau_n^+(1)-1}(H_n(f)),\lambda_{2\tau_n^-(1)}(H_n(f)),\lambda_{2\tau_n^+(2)-1}(H_n(f)),\lambda_{2\tau_n^-(2)}(H_n(f)),\dots\},$$

$$\{\lambda_1(T_n(f)),\dots,\lambda_n(T_n(f))\}=\{\lambda_{2\tau_n^+(1)-1}(T_n(f)),\lambda_{2\tau_n^-(1)}(T_n(f)),\lambda_{2\tau_n^+(2)-1}(T_n(f)),\lambda_{2\tau_n^-(2)}(T_n(f)),\dots\},$$

where τ_n^+ and τ_n^- are two permutations for which the minima in (3.17)–(3.18) are attained. Then, (3.17)–(3.18) imply

$$\max_{i=1,\dots,n}|f(x_{i,n})-\lambda_i(T_n(f))|\to 0 \text{ as } n\to\infty,$$

which yields (3.12). Moreover, after the above rearrangement of the eigenvalues of $T_n(f)$ and $H_n(f)$, (2.4)–(2.7) continue to hold, because, by construction, the considered rearrangement is associated with a permutation τ_n of the columns of V_n and the eigenvalues of $T_n(f)$ and $H_n(f)$ such that τ_n maps odd indices to odd indices and even indices to even indices. Of course, (2.4)–(2.7) continue to hold with a new matrix V_n obtained by permuting the columns of the old V_n through the permutation τ_n. With abuse of notation, we denote again by V_n the new matrix V_n, so that (2.4)–(2.7) hold unchanged. The thesis is proved. □

Theorem 3.3 is our third main result. It shows, among others, that, for a real even bounded function f with a finite number of local maximum/minimum points and discontinuities, the alternating sign relationship (3.11) holds for a suitable a.u. grid $\{x_{i,n}\}_{i=1,\dots,n}$ with $\varepsilon_{i,n}=0$ for every $i=1,\dots,n$ and every n.

Theorem 3.3. *Let $f:[-\pi,\pi]\to\mathbb{R}$ be even and bounded with a finite number of local maximum points, local minimum points, and discontinuity points, and with $\mathcal{ER}(f)=[\inf_{[0,\pi]}f,\sup_{[0,\pi]}f]$ and $(\inf_{[0,\pi]}f,\sup_{[0,\pi]}f)\subseteq f([0,\pi])$. Then, for every n, there exist an a.u. grid $\{x_{i,n}\}_{i=1,\dots,n}$ in $[0,\pi]$ and a real unitary matrix V_n such that (2.4)–(2.7) hold and*

$$\lambda_i(T_n(f))=f(x_{i,n}),\qquad i=1,\dots,n. \tag{3.19}$$

Proof. By Theorem 3.1, for every n there exists a real unitary matrix V_n such that (2.4)–(2.7) are satisfied and

$$\{\Lambda_n^+\}_n\sim f,\qquad \Lambda_n^+=\{\lambda_{2i-1}(H_n(f))\}_{i=1,\dots,\lceil n/2\rceil}=\{\lambda_{2i-1}(T_n(f))\}_{i=1,\dots,\lceil n/2\rceil}, \tag{3.20}$$

$$\{\Lambda_n^-\}_n\sim -f,\qquad \Lambda_n^-=\{\lambda_{2i}(H_n(f))\}_{i=1,\dots,\lfloor n/2\rfloor}=\{-\lambda_{2i}(T_n(f))\}_{i=1,\dots,\lfloor n/2\rfloor}. \tag{3.21}$$

Note that if we permute the columns of V_n and the eigenvalues of $T_n(f)$ and $H_n(f)$ through the same permutation τ of $\{1,\dots,n\}$ such that τ maps odd indices to odd indices and even indices to even indices, then (2.4)–(2.7) continue to hold.

By (3.20)–(3.21), Theorem 2.4, and the assumptions on f, the hypotheses of Theorem 2.2 are satisfied for f and Λ_n^+ as well as for $-f$ and Λ_n^-. Hence, by Theorem 2.2 applied first with f and Λ_n^+ and then with $-f$ and Λ_n^-, we infer the existence of two a.u. grids $\{x_{i,n}^+\}_{i=1,\dots,\lceil n/2\rceil}$, $\{x_{i,n}^-\}_{i=1,\dots,\lfloor n/2\rfloor}$ in $[0,\pi]$ and two permutations τ_n^+ of $\{1,\dots,\lceil n/2\rceil\}$ and τ_n^- of $\{1,\dots,\lfloor n/2\rfloor\}$ such that, for every n,

$$\lambda_{2\tau_n^+(i)-1}(T_n(f))=f(x_{i,n}^+),\qquad i=1,\dots,\lceil n/2\rceil, \tag{3.22}$$

$$-\lambda_{2\tau_n^-(i)}(T_n(f))=-f(x_{i,n}^-),\qquad i=1,\dots,\lfloor n/2\rfloor. \tag{3.23}$$

Define

$$\{x_{1,n},\dots,x_{n,n}\}=\{x_{1,n}^+,x_{1,n}^-,x_{2,n}^+,x_{2,n}^-,\dots\}$$

and rearrange the eigenvalues of $H_n(f)$ and $T_n(f)$ as follows:

$$\{\lambda_1(H_n(f)),\dots,\lambda_n(H_n(f))\}=\{\lambda_{2\tau_n^+(1)-1}(H_n(f)),\lambda_{2\tau_n^-(1)}(H_n(f)),\lambda_{2\tau_n^+(2)-1}(H_n(f)),\lambda_{2\tau_n^-(2)}(H_n(f)),\dots\},$$

$$\{\lambda_1(T_n(f)),\dots,\lambda_n(T_n(f))\}=\{\lambda_{2\tau_n^+(1)-1}(T_n(f)),\lambda_{2\tau_n^-(1)}(T_n(f)),\lambda_{2\tau_n^+(2)-1}(T_n(f)),\lambda_{2\tau_n^-(2)}(T_n(f)),\dots\}.$$

Note that $\{x_{i,n}\}_{i=1,\dots,n}$ is an a.u. grid in $[0,\pi]$ by Lemma 3.1. Moreover, after the above rearrangement of the eigenvalues of $T_n(f)$ and $H_n(f)$, by (3.22)–(3.23) we have

$$\lambda_i(T_n(f))=f(x_{i,n}),\qquad i=1,\dots,n,$$

which is (3.19). In addition, (2.4)–(2.7) continue to hold, because, by construction, the considered rearrangement is associated with a permutation τ_n of the columns of V_n and the eigenvalues of $T_n(f)$ and $H_n(f)$ such that τ_n maps odd indices to odd indices and even indices to even indices. Of course, (2.4)–(2.7) continue to hold with a new matrix V_n obtained by permuting the columns of the old V_n through the permutation τ_n. With abuse of notation, we denote again by V_n the new matrix V_n, so that (2.4)–(2.7) hold unchanged. The thesis is proved. □

Theorem 3.4 is our fourth main result. It is an extension of Corollary 2.1 to the case where $f \in L^1([-\pi,\pi])$ is only assumed to have real Fourier coefficients. In this case, the moduli of the eigenvalues of $H_n(f)$ coincide with the singular values of $T_n(f)$, as shown by the following remark.

Remark 3.1. For every matrix $A \in \mathbb{C}^{n\times n}$, the singular values of A and Y_nA coincide because Y_n is a unitary (permutation) matrix. In particular, for every $f \in L^1([-\pi,\pi])$, the singular values of $T_n(f)$ and $H_n(f)$ coincide. In the case where $T_n(f)$ is real, which happens whenever the Fourier coefficients of f are real, the matrix $H_n(f)$ is real and symmetric, and so the singular values of $H_n(f)$, i.e., the singular values of $T_n(f)$, coincide with the moduli of the eigenvalues of $H_n(f)$.

Theorem 3.4. *Suppose that the Fourier coefficients of $f \in L^1([-\pi,\pi])$ are real. For every n, let*

$$T_n(f) = U_n\Sigma_nV_n^*, \qquad \Sigma_n = \operatorname*{diag}_{i=1,\dots,n} \sigma_i(T_n(f)),$$

be a singular value decomposition of $T_n(f)$. Then, for every n, we can arrange the eigenvalues of $H_n(f)$ so that

$$|\lambda_i(H_n(f))| = \sigma_i(T_n(f)), \qquad i=1,\dots,n, \tag{3.24}$$

$$H_n(f)^2 = V_n\Sigma_n^2V_n^*, \tag{3.25}$$

$$H_n(f)^2 = Y_nU_n\Sigma_n^2(Y_nU_n)^*. \tag{3.26}$$

In particular, the columns of V_n (right singular vectors of $T_n(f)$) and the columns of Y_nU_n (flipped left singular vectors of $T_n(f)$) are orthonormal bases of $\mathbb{C}^n$ consisting of eigenvectors of $H_n(f)^2$.

Proof. This proof of Theorem 3.4 is simpler than the original one proposed by the authors and is due to an anonymous reviewer. We know from Remark 3.1 that the eigenvalues of $H_n(f)$ can be arranged so that (3.24) holds. (3.25)–(3.26) follow from (3.24) by taking into account the singular value decomposition $T_n(f) = U_n\Sigma_nV_n^*$ and the fact that $H_n(f) = Y_nT_n(f)$ is real and symmetric:

$$H_n(f)^2 = H_n(f)^*H_n(f) = T_n(f)^*T_n(f) = V_n\Sigma_n^2V_n^*,$$

$$H_n(f)^2 = H_n(f)H_n(f)^* = Y_nT_n(f)T_n(f)^*Y_n = Y_nU_n\Sigma_n^2(Y_nU_n)^*. \quad \square$$

Remark 3.2. For every diagonalizable matrix $A \in \mathbb{C}^{n\times n}$, the eigenvectors of A and A^2 coincide whenever the eigenvalues of A^2 (i.e., the squares of the eigenvalues of A) are distinct. More precisely, in this case we have that $\boldsymbol{v}$ is an eigenvector of A associated with the eigenvalue λ if and only if $\boldsymbol{v}$ is an eigenvector of A^2 associated with λ^2. It follows that, *in Theorem 3.4, we can replace (3.25) with $H_n(f) = V_n\left[\mathrm{diag}_{i=1,\dots,n}\,\lambda_i(H_n(f))\right]V_n^*$ and (3.26) with $H_n(f) = Y_nU_n\left[\mathrm{diag}_{i=1,\dots,n}\,\lambda_i(H_n(f))\right](Y_nU_n)^*$ whenever the eigenvalues of $H_n(f)^2$ are distinct.*

Theorem 3.5 is our fifth main result. It provides localization results for the eigenvalues of $H_n(f)$ under different assumptions on the function f. To prove Theorem 3.5, we need two auxiliary lemmas. The first one is a plain extension of Widom's Lemma 2.2. The second one combines a few classical results on Toeplitz matrices [13].

Lemma 3.2. *Let $f \in L^1([-\pi,\pi])$ and let d_f be the distance of 0 from $\mathrm{Co}(\mathcal{ER}(f))$ in $\mathbb{C}$. Suppose that $|f|$ is not a.e. constant. Then, the singular values of $T_n(f)$ lie in $[d_f, M_{|f|})$ for all n, where $M_{|f|} = \operatorname{ess\,sup}_{[-\pi,\pi]}|f|$.*

Proof. By Widom's Lemma 2.2, the singular values of $T_n(f)$ lie in $[d_f, M_{|f|}]$ for all n. We show that no singular value of $T_n(f)$ can be equal to $M_{|f|}$, i.e., $\|T_n(f)\| < M_{|f|}$, where $\|T_n(f)\|$ is the spectral (or Euclidean) norm of $T_n(f)$ (the largest singular value of $T_n(f)$). It is known that $\|T_n(f)\| \le \|T_n(|f|)\|$; see, e.g., [13, Lemma 6.3] applied with $p=\infty$. By Theorem 2.4 and the assumption that $|f|$ is not a.e. constant, we infer that $T_n(|f|)$ is a Hermitian positive definite matrix whose eigenvalues lie in $(m_{|f|}, M_{|f|}) \subseteq (0, M_{|f|})$. In particular, the largest eigenvalue of $T_n(|f|)$ coincides with $\|T_n(|f|)\|$ and is smaller than $M_{|f|}$. Thus, $\|T_n(f)\| \le \|T_n(|f|)\| < M_{|f|}$. □

Lemma 3.3. *Let $f(\theta) = \sum_{k=-r}^{r} f_k e^{\mathrm{i}k\theta}$ be a trigonometric polynomial of degree r, and let*

$$m_{|f|} = \min_{[-\pi,\pi]}|f|, \qquad M_{|f|} = \max_{[-\pi,\pi]}|f|.$$

Then, for every n, the singular values of $T_n(f)$ lie in $[m_{|f|}, M_{|f|}]$ except for at most $2r$ outliers.

Proof. The thesis is obvious for $n \le 2r$. Suppose that $n \ge 2r+1$. Let $C_n(f)$ be the circulant matrix defined in [13, p. 109]. By the first inequality in [13, p. 110], we have

$$\mathrm{rank}(T_n(f) - C_n(f)) \le 2r.$$

Hence, by the interlacing theorem for singular values [13, Theorem 2.11], the singular values of $T_n(f)$ lie between $\sigma_{\min}(C_n(f))$ and $\sigma_{\max}(C_n(f))$, except for at most $2r$ singular values smaller than $\sigma_{\min}(C_n(f))$ and $2r$ singular values larger than $\sigma_{\max}(C_n(f))$. The singular values of $T_n(f)$ larger than $\sigma_{\max}(C_n(f))$, if any, are anyway $\le M_{|f|}$ by Lemma 2.2. Moreover, we know from [13, Theorem 6.4] that $\sigma_{\min}(C_n(f))$ and $\sigma_{\max}(C_n(f))$ lie between $[m_{|f|}, M_{|f|}]$. Thus, all the singular values of $T_n(f)$ lie in $[m_{|f|}, M_{|f|}]$ except for at most $2r$ outliers (smaller than $m_{|f|}$). □

Theorem 3.5. *Suppose that the Fourier coefficients of $f \in L^1([-\pi,\pi])$ are real, and let*

$$d_f = \text{distance of } 0 \text{ from } \mathrm{Co}(\mathcal{ER}(f)) \text{ in } \mathbb{C},$$

$$m_{|f|} = \operatorname*{ess\,inf}_{[-\pi,\pi]} |f|, \qquad M_{|f|} = \operatorname*{ess\,sup}_{[-\pi,\pi]} |f|.$$

Then, for every n, the following properties hold.

1. *The eigenvalues of $H_n(f)$ lie in $[-M_{|f|}, -d_f] \cup [d_f, M_{|f|}]$.*
2. *Assume that $|f|$ is not a.e. constant. Then, the eigenvalues of $H_n(f)$ lie in $(-M_{|f|}, -d_f] \cup [d_f, M_{|f|})$.*
3. *Assume that $|f| \ne m_{|f|}$ a.e. and $|f| \ne M_{|f|}$ a.e. Then, the eigenvalues of $H_n(f)$ lie in $(-M_{|f|}, -m_{|f|}] \cup [m_{|f|}, M_{|f|})$ except for at most $o(n)$ outliers lying in $(-m_{|f|}, m_{|f|})$.*
4. *Assume that $f(\theta) = \sum_{k=-r}^{r} f_k \mathrm{e}^{\mathrm{i}k\theta}$ is a trigonometric polynomial of degree r. Then, the eigenvalues of $H_n(f)$ lie in $[-M_{|f|}, -m_{|f|}] \cup [m_{|f|}, M_{|f|}]$ except for at most $2r$ outliers lying in $(-m_{|f|}, m_{|f|})$.*
5. *Assume that f belongs to the Krein algebra. Then, for every $\varepsilon > 0$ there exists a constant C_ε such that the eigenvalues of $H_n(f)$ lie in $[-M_{|f|}, -m_{|f|} + \varepsilon) \cup (m_{|f|} - \varepsilon, M_{|f|}]$ except for at most C_ε outliers.*

Proof. We know that $|\lambda_i(H_n(f))| = \sigma_i(T_n(f))$, $i = 1, \ldots, n$. Hence, item 1 follows from Lemma 2.2; item 2 follows from Lemma 3.2; item 3 follows from item 2 and the asymptotic singular value distribution $\{T_n(f)\}_n \sim_\sigma f$ in Theorem 2.3, which implies that the number of singular values of $T_n(f)$ lying outside $\mathcal{ER}(|f|) \subseteq [m_{|f|}, M_{|f|}]$ is $o(n)$ by [3, Theorem 4.7]; item 4 follows from item 1 and Lemma 3.3; and, finally, item 5 follows from item 1 and Lemma 2.3. □

Our last main result (Theorem 3.6) is more an observation than a "main result", but we decided anyway to state it here, in the section of main results, as it completes our spectral study of flipped Toeplitz matrices. To prove Theorem 3.6, we need the following basic lemmas, which can be seen as corollaries of [15, Proposition 3.1.2]; see also [30, Exercise 4.5]. For the reader's convenience, we include the short proofs.

Lemma 3.4. *Let $f \in L^1([-\pi,\pi])$. Then, the following are equivalent.*

1. *f is real and even a.e. in $[-\pi,\pi]$, i.e., $f(-x) = f(x) \in \mathbb{R}$ for a.e. $x \in [-\pi,\pi]$.*
2. *The Fourier coefficients of f are real and even, i.e., $f_{-k} = f_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$.*

Proof. ($1 \implies 2$) Suppose that $f(-x) = f(x) \in \mathbb{R}$ for almost every $x \in [-\pi,\pi]$. Then, for every $k \in \mathbb{Z}$,

$$f_{-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t = f_k$$

and f_k is real, because

$$f_k = \frac{1}{2}(f_{-k} + f_k) = \frac{1}{2}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x + \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos(kx)\mathrm{d}x.$$

($2 \implies 1$) Suppose that the Fourier coefficients of f are real and even. In order to prove that f is real and even a.e. in $[-\pi,\pi]$, it suffices to prove that the three functions $f(-x)$, $f(x)$, $\overline{f(x)}$ have the same Fourier coefficients, which means that they coincide a.e. [27, Theorem 5.15]. Let $\{a_k\}_{k\in\mathbb{Z}}$ (resp., $\{b_k\}_{k\in\mathbb{Z}}$, $\{f_k\}_{k\in\mathbb{Z}}$) be the sequence of Fourier coefficients of $f(-x)$ (resp., $\overline{f(x)}$, $f(x)$). Then, taking into account that the Fourier coefficients of f are real and even, for every $k \in \mathbb{Z}$ we have

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x} = \overline{f_{-k}} = f_{-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t = a_k,$$

hence $b_k = a_k = f_{-k} = f_k$. □

Lemma 3.5. *Suppose that the Fourier coefficients of $f \in L^1([-\pi,\pi])$ are real. Then, $f(-x) = \overline{f(x)}$ for almost every $x \in [-\pi,\pi]$.*

Proof. It suffices to prove that the two functions $f(-x)$ and $\overline{f(x)}$ have the same Fourier coefficients, which means that they coincide a.e. [27, Theorem 5.15]. Let $\{a_k\}_{k\in\mathbb{Z}}$ (resp., $\{b_k\}_{k\in\mathbb{Z}}$, $\{f_k\}_{k\in\mathbb{Z}}$) be the sequence of Fourier coefficients of $f(-x)$ (resp., $\overline{f(x)}$, $f(x)$). Then, taking into account that the Fourier coefficients of f are real, for every $k \in \mathbb{Z}$ we have

$$b_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{f(x)}\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x = \overline{\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x} = \overline{f_{-k}} = f_{-k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\mathrm{e}^{\mathrm{i}kx}\mathrm{d}x = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(-t)\mathrm{e}^{-\mathrm{i}kt}\mathrm{d}t = a_k,$$

hence $b_k = a_k$. □

To simplify the statement of Theorem 3.6, we borrow a notation from [9]: for every $g : [0,\pi] \to \mathbb{C}$, we define the function $\psi_g : [0,2\pi] \to \mathbb{C}$ by setting

$$\psi_g(x) = \begin{cases} g(x), & x \in [0,\pi], \\ -g(x-\pi), & x \in (\pi, 2\pi]. \end{cases}$$

Moreover, for every $f : [-\pi,\pi] \to \mathbb{C}$, we denote by $f|_{[0,\pi]}$ the restriction of f to the interval $[0,\pi]$.

Theorem 3.6. *Let $f \in L^1([-\pi,\pi])$. Then, the following properties hold.*

1. *Suppose that the Fourier coefficients of f are real. Then,*

$$\{T_n(f)\}_n \sim_\sigma f|_{[0,\pi]}, \qquad \{H_n(f)\}_n \sim_\sigma f|_{[0,\pi]}.$$

2. *Suppose that the Fourier coefficients of f are real and even. Then,*

$$\{T_n(f)\}_n \sim_\lambda f|_{[0,\pi]}, \qquad \{H_n(f)\}_n \sim_\lambda \psi_{|f|_{[0,\pi]}|}, \qquad \{H_n(f)\}_n \sim_\lambda \psi_{f|_{[0,\pi]}}.$$

Proof. 1. By Lemma 3.5, we have $f(-x) = \overline{f(x)}$ for almost every $x \in [-\pi,\pi]$, hence $|f(-x)| = |f(x)|$ for almost every $x \in [-\pi,\pi]$. Thus, the relation $\{T_n(f)\}_n \sim_\sigma f|_{[0,\pi]}$ is a consequence of the relation $\{T_n(f)\}_n \sim_\sigma f$ (which holds by Theorem 2.4) and the definition of asymptotic singular value distribution. The relation $\{H_n(f)\}_n \sim_\sigma f|_{[0,\pi]}$ follows immediately from $\{T_n(f)\}_n \sim_\sigma f|_{[0,\pi]}$ and the fact that the singular values of $H_n(f)$ and $T_n(f)$ coincide; see Remark 3.1.

2. By Lemma 3.4, f is real and even a.e. Thus, the relation $\{T_n(f)\}_n \sim_\lambda f|_{[0,\pi]}$ is a consequence of the relation $\{T_n(f)\}_n \sim_\lambda f$ (which holds by Theorem 2.4) and the definition of asymptotic spectral distribution. The relation $\{H_n(f)\}_n \sim_\lambda \psi_{|f|_{[0,\pi]}|}$ is a consequence of the relation

$$\{H_n(f)\}_n \sim_\lambda \mathrm{diag}(|f(x)|, -|f(x)|), \qquad x \in [0,\pi], \tag{3.27}$$

(which holds by Theorem 2.5 and the evenness of f) and Lemma 2.1 applied with $[\alpha,\beta] = [0,2\pi]$. Finally, the relation $\{H_n(f)\}_n \sim_\lambda \psi_{f|_{[0,\pi]}}$ is a consequence of the relation $\{H_n(f)\}_n \sim_\lambda \psi_{|f|_{[0,\pi]}|}$ and the definition of asymptotic spectral distribution, taking into account the f is real a.e. □

4. Numerical experiments

We present in Examples 4.1–4.3 a few numerical experiments that illustrate Theorems 3.2 and 3.3. Note that the thesis of Theorem 3.3 is stronger than the thesis of Theorem 3.2, because the existence of an a.u. grid $\{x_{i,n}\}_{i=1,\ldots,n}$ in $[0,\pi]$ such that (3.19) is satisfied implies that (3.12) is satisfied for every a.u. grid $\{x_{i,n}\}_{i=1,\ldots,n}$ in $[0,\pi]$. Thus, for functions f satisfying the hypotheses of both Theorems 3.2 and 3.3, we just illustrate Theorem 3.3.

Example 4.1. Let $f : [-\pi,\pi] \to \mathbb{R}$,

$$f(\theta) = \begin{cases} 1, & 0 \le \theta < \pi/2, \\ \theta + 1 - \pi/2, & \pi/2 \le \theta \le \pi, \\ f(-\theta), & -\pi \le \theta < 0. \end{cases} \tag{4.1}$$

Fig. 4.1 shows the graph of f over the interval $[0,\pi]$. The function f satisfies the hypotheses of Theorem 3.2. Hence, by Theorem 3.2, for every n and every a.u. grid $\{x_{i,n}\}_{i=1,\ldots,n}$ in $[0,\pi]$, there exists a real unitary matrix V_n such that (2.4)–(2.7) hold and

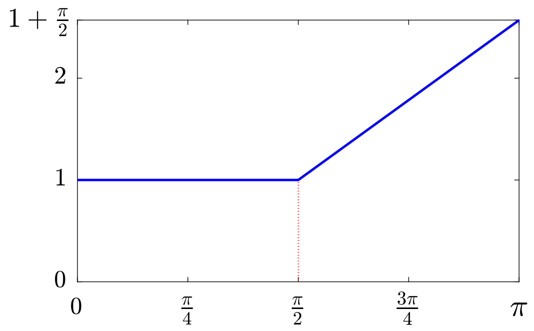

Fig. 4.1. Example 4.1: Graph on the interval $[0,\pi]$ of the function $f(\theta)$ defined in (4.1).

Table 4.1
Example 4.1: Computation of M_n for increasing values of n.

n	M_n
8	0.0851
16	0.0632
32	0.0454
64	0.0312
128	0.0206
256	0.0132
512	0.0082
1024	0.0050

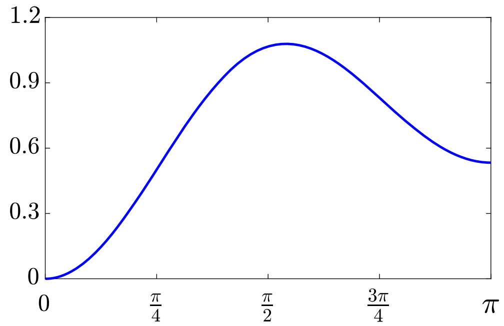

Fig. 4.2. Example 4.2: Graph on the interval $[0,\pi]$ of the function $f(\theta)$ defined in (4.3).

Table 4.2
Example 4.2: Computation of $m(\mathcal{G}_n)$ for increasing values of n.

n	$m(\mathcal{G}_n)$
8	0.2475
16	0.1304
32	0.0753
64	0.0403
128	0.0328
256	0.0245
512	0.0119
1024	0.0060

$$M_n = \max_{i=1,\ldots,n} |f(x_{i,n}) - \lambda_i(T_n(f))| \to 0 \text{ as } n\to\infty. \tag{4.2}$$

To provide numerical evidence of this, for the values of n considered in Table 4.1, we arranged the eigenvalues of $T_n(f)$ so that (2.4)–(2.7) are satisfied. In other words, we arranged the eigenvalues of $T_n(f)$ so that the eigenvector associated with the ith eigenvalue $\lambda_i(T_n(f))$ is either symmetric or skew-symmetric depending on whether i is odd or even. Then, we computed M_n in the case of the a.u. grid $x_{i,n}=\frac{i\pi}{n+1}$, $i=1,\ldots,n$. We see from the table that $M_n\to 0$ as $n\to\infty$, though the convergence is slow.

Now we observe that f does not satisfy the hypotheses of Theorem 3.3. Actually, f satisfies all the hypotheses of Theorem 3.3 except for the assumption that f has a finite number of local maximum/minimum points. Indeed, f is constant on $[0,\pi/2]$ and so all points in $[0,\pi/2)$ are both local maximum and local minimum points for f according to our Definition 2.3. We observe that, in fact, the thesis of Theorem 3.3 does not hold in this case, because there is no a.u. grid $\{x_{i,n}\}_{i=1,\ldots,n}$ in $[0,\pi]$ such that, for every n,

$$\lambda_i(T_n(f)) = f(x_{i,n}), \qquad i=1,\ldots,n,$$

for a suitable ordering of the eigenvalues of $T_n(f)$. Indeed, the eigenvalues of $T_n(f)$ are contained in $(1,1+\pi/2)$ by Theorem 2.4 and so any grid $\{x_{i,n}\}_{i=1,\ldots,n}\subset[0,\pi]$ satisfying the previous condition must be contained in $(\pi/2,\pi)$, which implies that it cannot be a.u. in $[0,\pi]$.

Example 4.2. This example is suggested by the cubic B-spline Galerkin discretization of second-order eigenvalue (and Poisson) problems [14, Section 2.4.1]. Let $f:[-\pi,\pi]\to\mathbb{R}$,

$$f(\theta) = \frac{2}{3} - \frac{1}{4}\cos(\theta) - \frac{2}{5}\cos(2\theta) - \frac{1}{60}\cos(3\theta). \tag{4.3}$$

Fig. 4.2 shows the graph of f over the interval $[0,\pi]$. The function f satisfies the hypotheses of Theorem 3.3. Hence, by Theorem 3.3, for every n there exist an a.u. grid $\{x_{i,n}\}_{i=1,\ldots,n}$ in $[0,\pi]$ and a real unitary matrix V_n such that (2.4)–(2.7) hold and

$$\lambda_i(T_n(f)) = f(x_{i,n}), \qquad i=1,\ldots,n. \tag{4.4}$$

To provide numerical evidence of this, for the values of n considered in Table 4.2, we arranged the eigenvalues of $T_n(f)$ so that (2.4)–(2.7) are satisfied. In other words, we arranged the eigenvalues of $T_n(f)$ so that the eigenvector associated with the ith eigenvalue $\lambda_i(T_n(f))$ is either symmetric or skew-symmetric depending on whether i is odd or even. Then, we computed a grid

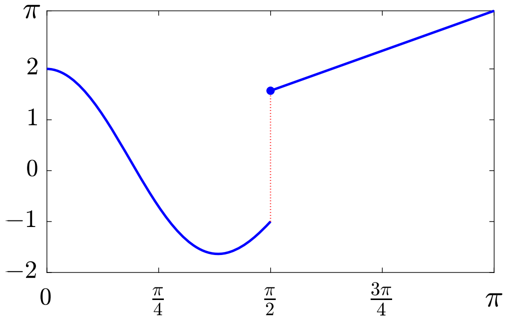

Fig. 4.3. Example 4.3: Graph on the interval $[0, \pi]$ of the function $f(\theta)$ defined in (4.5).

Table 4.3
Example 4.3: Computation of $m(\mathcal{G}_n)$ for increasing values of n.

n	$m(\mathcal{G}_n)$
8	0.5771
16	0.4633
32	0.2421
64	0.2082
128	0.1127
256	0.0812
512	0.0336
1024	0.0183

$\mathcal{G}_n = \{x_{i,n}\}_{i=1,\dots,n}$ satisfying (4.4) and we reported in Table 4.2 its uniformity measure $m(\mathcal{G}_n)$. We see from the table that $m(\mathcal{G}_n) \to 0$ as $n \to \infty$, meaning that $\mathcal{G}_n$ is a.u. in $[0, \pi]$, though the convergence to 0 of $m(\mathcal{G}_n)$ is slow.

Example 4.3. In this example, we consider a discontinuous function. Let $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(\theta) = \begin{cases} \cos(2\theta) + \cos(3\theta), & 0 \le \theta < \pi/2, \\ \theta, & \pi/2 \le \theta \le \pi, \\ f(-\theta), & -\pi \le \theta < 0. \end{cases} \tag{4.5}$$

Fig. 4.3 shows the graph of f over the interval $[0, \pi]$. The function f satisfies the hypotheses of Theorem 3.3 just as the function f of Example 4.2. Hence, we proceeded exactly as in Example 4.2. The results are collected in Table 4.3, which is the version of Table 4.2 for this example.

In Examples 4.4–4.6, we show how the spectral localization results in Theorem 3.5 can be used to predict the MINRES performance on linear systems with coefficient matrix $H_n(f)$. We recall that the interest behind solving linear systems with coefficient matrix $H_n(f)$ lies in the fact that solving $H_n(f)\boldsymbol{x} = Y_n\boldsymbol{b}$ is equivalent to solving the Toeplitz linear system $T_n(f)\boldsymbol{x} = \boldsymbol{b}$. We also recall that the convergence bounds for CG and MINRES are formally identical and rely only on the eigenvalues of the system matrix; cf. [8, Eqs. (2.12) and (6.50)]. In particular, a fast convergence of both methods is usually observed when the eigenvalues of the system matrix are clustered at a small subset of $\mathbb{R}$ bounded away from 0. For more details on this subject, see [8, Sections 2.1.1 and 2.2] for CG and [8, Sections 6.1 and 6.2.4] for MINRES. In what follows, we denote by $\mathbf{1}_n$ the vector of all ones in $\mathbb{R}^n$.

Example 4.4. Let $f : [-\pi, \pi] \to \mathbb{C}$,

$$f(\theta) = -e^{i\theta} + \frac{3}{2} + e^{-i\theta} + e^{-2i\theta} + e^{-3i\theta}. \tag{4.6}$$

The Fourier coefficients of f are real as in the hypotheses of Theorem 3.5. A direct computation shows that the values d_f, $m_{|f|}$, $M_{|f|}$ in the statement of Theorem 3.5 are given by

$$d_f = 0, \qquad m_{|f|} \approx 1.11, \qquad M_{|f|} \approx 3.53.$$

In particular, 0 lies inside the convex hull $\mathrm{Co}(\mathcal{ER}(f))$; see also Fig. 4.4. By items 2 and 4 in Theorem 3.5, for every n, the eigenvalues of $H_n(f)$ are contained in $(-3.53, 3.53)$ and, moreover, at most 6 eigenvalues can lie outside $(-3.53, -1.11] \cup [1.11, 3.53)$. Therefore, even if $d_f = 0$, we can predict a fast convergence of MINRES for linear systems with coefficient matrix $H_n(f)$, due to the clustering of the eigenvalues of $H_n(f)$ at $(-3.53, -1.11] \cup [1.11, 3.53)$, which is the union of two intervals sufficiently bounded away from 0. This prediction is confirmed by Table 4.4, which shows that the number of MINRES iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6} is essentially independent of the matrix size n. In this sense, the MINRES convergence is optimal with respect to n. Table 4.4 also reports for each n the number of outliers, i.e., the number of eigenvalues of $H_n(f)$ lying outside $(-3.53, -1.11] \cup [1.11, 3.53)$. As expected, the number of outliers is smaller than 6 (actually, equal to 0) for all the considered values of n.

Example 4.5. Let $f : [-\pi, \pi] \to \mathbb{C}$,

$$f(\theta) = -\frac{1}{10} + e^{-i\theta/2}. \tag{4.7}$$

The Fourier coefficients of f are real as in the hypotheses of Theorem 3.5; they are explicitly given by

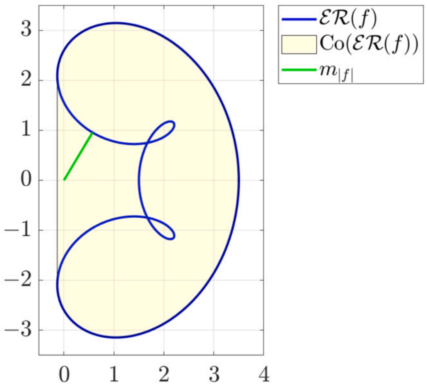

Fig. 4.4. Example 4.4: (Essential) range of the function $f(\theta)$ defined in (4.6) and its convex hull. The value $m_{|f|}$ is also indicated.

Table 4.4
Example 4.4: Number of outliers and number of MINRES iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6} for increasing values of n.

n	Outliers	MINRES iterations
32	0	28
64	0	35
128	0	37
256	0	35
512	0	35
1024	0	33
2048	0	33
4096	0	31

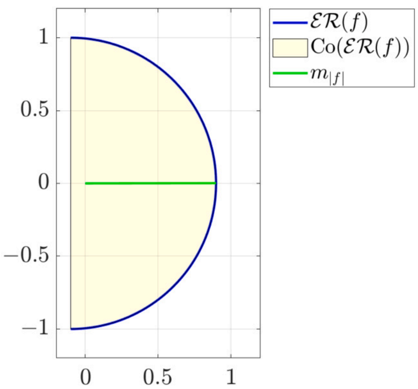

Fig. 4.5. Example 4.5: (Essential) range of the function $f(\theta)$ defined in (4.7) and its convex hull. The value $m_{|f|}$ is also indicated.

Table 4.5
Example 4.5: Number of outliers and number of MINRES iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6} for increasing values of n.

n	Outliers	MINRES iterations
32	2	11
64	2	10
128	3	11
256	3	11
512	3	10
1024	3	10
2048	3	10
4096	4	12

$$f_k = \begin{cases} -\dfrac{1}{10} + \dfrac{2}{\pi}, & \text{if } k = 0, \\[2mm] \dfrac{2(-1)^k}{\pi(2k+1)}, & \text{if } k \neq 0. \end{cases}$$

A direct computation shows that the values d_f, $m_{|f|}$, $M_{|f|}$ in the statement of Theorem 3.5 are given by

$$d_f = 0, \qquad m_{|f|} = 0.9, \qquad M_{|f|} \approx 1.00.$$

In particular, 0 lies inside the convex hull $\mathrm{Co}(\mathcal{ER}(f))$; see also Fig. 4.5. Note that f does not belong to the Krein algebra, because

$$\sum_{k\in\mathbb{Z}} |k|\,|f_k|^2 = \sum_{k\in\mathbb{Z}} \frac{4|k|}{\pi^2(2k+1)^2} = \infty.$$

In particular, we are not in the hypotheses of item 5 of Theorem 3.5. Nevertheless, we are in the hypotheses of item 3 of Theorem 3.5, because

$$|f(\theta)| = \frac{1}{10}\sqrt{101 - 2\cos\left(\frac{\theta}{2}\right)}$$

cannot be equal to $m_{|f|}$ or $M_{|f|}$ on a set of positive measure. By items 2 and 3 in Theorem 3.5, for every n, the eigenvalues of $H_n(f)$ are contained in $(-1.00, 1.00)$ and, moreover, the number of eigenvalues lying outside $(-1.00, -0.9] \cup [0.9, 1.00)$ is at most $o(n)$. Therefore, even if $d_f = 0$, we can predict a fast convergence of MINRES for linear systems with coefficient matrix $H_n(f)$, due to the clustering of the eigenvalues of $H_n(f)$ at $(-1.00, -0.9] \cup [0.9, 1.00)$, which is the union of two intervals sufficiently bounded away from 0. This prediction is confirmed by Table 4.5, which shows that the number of MINRES iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6} is essentially independent of the matrix size n. In this sense, the MINRES convergence is optimal with respect to n. Table 4.5 also reports for each n the number of outliers, i.e., the number of eigenvalues of $H_n(f)$ lying outside $(-1.00, -0.9] \cup [0.9, 1.00)$. As expected, the number of outliers is small compared to the matrix size n for all the considered values of n.

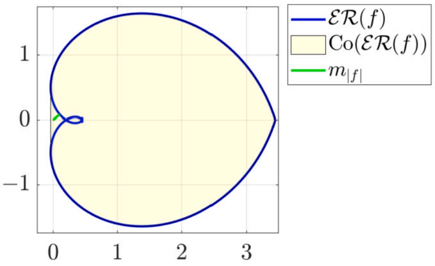

Fig. 4.6. Example 4.6: (Essential) range of the function $f(\theta)$ defined in (4.8) and its convex hull. The value $m_{|f|}$ is also indicated.

Table 4.6
Example 4.6: Number of outliers and number of MINRES and PMINRES iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6} for increasing values of n.

n	Outliers	MINRES iterations	PMINRES iterations
32	0	31	14
64	0	62	13
128	0	118	13
256	0	163	11
512	0	225	11
1024	0	237	11
2048	0	233	11
4096	0	225	10

Example 4.6. Let $f : [-\pi, \pi] \to \mathbb{C}$,

$$f(\theta) = \frac{9}{20} + \frac{\theta^2}{\pi^2}(-\mathrm{e}^{\mathrm{i}\theta} + \mathrm{e}^{2\mathrm{i}\theta} - \mathrm{e}^{3\mathrm{i}\theta}). \tag{4.8}$$

The Fourier coefficients of f are real as in the hypotheses of Theorem 3.5; they are explicitly given by

$$f_k = \begin{cases} \dfrac{9}{20} + \dfrac{49}{18\pi^2}, & \text{if } k = 0, \\ -\dfrac{1}{3} - \dfrac{5}{2\pi^2}, & \text{if } k = 1, 3, \\ \dfrac{1}{3} + \dfrac{4}{\pi^2}, & \text{if } k = 2, \\ \dfrac{6(-1)^k\left(k^4 - 8k^3 + 24k^2 - 32k + \dfrac{49}{3}\right)}{\pi^2(k-1)^2(k-2)^2(k-3)^2}, & \text{if } k \neq 0, 1, 2, 3. \end{cases}$$

A direct computation shows that the values d_f, $m_{|f|}$, $M_{|f|}$ in the statement of Theorem 3.5 are given by

$$d_f = 0, \qquad m_{|f|} \approx 0.13, \qquad M_{|f|} = 3.45.$$

In particular, 0 lies inside the convex hull $\mathrm{Co}(\mathcal{ER}(f))$; see also Fig. 4.6. Note that f does not belong to the Krein algebra, because

$$\sum_{k\in\mathbb{Z}} |k|\,|f_k|^2 = |f_1|^2 + 2|f_2|^2 + 3|f_3|^2 + \sum_{k\in\mathbb{Z}\setminus\{0,1,2,3\}} \frac{6\,|k|\left|k^4 - 8k^3 + 24k^2 - 32k + \dfrac{49}{3}\right|}{\pi^2(k-1)^2(k-2)^2(k-3)^2} = \infty.$$

In particular, we are not in the hypotheses of item 5 of Theorem 3.5. Nevertheless, we are in the hypotheses of item 3 of Theorem 3.5, because

$$|f(\theta)| = \frac{\sqrt{81\pi^4 + 1200\,\theta^4 - (360\pi^2 + 1600\,\theta^2)\,\theta^2\cos(\theta) + (360\pi^2 + 800\,\theta^2)\,\theta^2\cos(2\theta) - 360\pi^2\theta^2\cos(3\theta)}}{20\pi^2}$$

cannot be equal to $m_{|f|}$ or $M_{|f|}$ on a set of positive measure. By items 2 and 3 in Theorem 3.5, for every n, the eigenvalues of $H_n(f)$ are contained in $(-3.45, 3.45)$ and, moreover, the number of eigenvalues lying outside $(-3.45, -0.13] \cup [0.13, 3.45)$ is at most $o(n)$. Therefore, even if $d_f = 0$, we can (in principle) predict a fast convergence of MINRES for linear systems with coefficient matrix $H_n(f)$, due to the clustering of the eigenvalues of $H_n(f)$ at $(-3.45, -0.13] \cup [0.13, 3.45)$. This prediction is confirmed by Table 4.6, which shows that the number of MINRES iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6} is bounded by a constant independent of the matrix size n. In this sense, the MINRES convergence is optimal with respect to n. Table 4.6 also reports for each n the number of outliers, i.e., the number of eigenvalues of $H_n(f)$ lying outside $(-3.45, -0.13] \cup [0.13, 3.45)$. As expected, the number of outliers is small (actually, equal to 0) for all the considered values of n.

The only unpleasant aspect of Table 4.6 is that the number of MINRES iterations, although optimal with respect to n, is not so small for large n. This is due to the fact that the clustering set for the eigenvalues of $H_n(f)$, i.e., $(-3.45, -0.13] \cup [0.13, 3.45)$, is not so bounded away from 0, because 0.13 is close to 0. In order to improve the number of MINRES iterations, we can use preconditioning. This is shown in the last column of Table 4.6, which reports the number of preconditioned MINRES (PMINRES) iterations for solving the linear system $H_n(f)\boldsymbol{x} = \mathbf{1}_n$ up to a precision of 10^{-6}. The chosen preconditioner is

$$|C_n| = (C_n^* C_n)^{1/2},$$

where C_n is the Frobenius optimal circulant preconditioner for $T_n(f)$ first introduced by Chan in [6]. The efficiency of the preconditioner $|C_n|$ observed in Table 4.6 could have been predicted on the basis of the clustering results in [26, Proposition 4.1] and [9, Theorem 3.6], which essentially say that the eigenvalues of $|C_n|^{-1} H_n(f)$ are clustered "around" the two points 1 and −1. However, a spectral localization for the eigenvalues of $|C_n|^{-1}H_n(f)$ analogous to Theorem 3.5 is not available in the literature and, consequently, a mathematically rigorous justification of the efficiency of the preconditioner $|C_n|$ is missing. Providing a spectral localization result for $|C_n|^{-1}H_n(f)$ analogous to Theorem 3.5 may represent an interesting topic for future research.

5. Conclusions

We have studied the spectral properties of flipped Toeplitz matrices of the form $H_n(f) = Y_nT_n(f)$, where $T_n(f)$ is the $n\times n$ Toeplitz matrix generated by f and Y_n is the exchange (flip) matrix in (1.2). Our spectral results are collected in Theorems 3.1–3.6. The spectral properties obtained in this paper can be used in the convergence analysis of MINRES for the solution of real non-symmetric Toeplitz linear systems of the form $T_n(f)\mathbf{x} = \mathbf{b}$ after pre-multiplication of both sides by Y_n, as suggested by Pestana and Wathen [26]. This has been illustrated through numerical experiments in Section 4. We conclude this paper by mentioning two possible future lines of research.

1. Extend the spectral localizations in Theorem 3.5 to preconditioned matrices of the form $P_n^{-1}H_n(f)$, where P_n is either the preconditioner $|C_n|$ in Example 4.6 or the Toeplitz matrix $T_n(|f|)$ generated by $|f|$. The first step in this direction could be the spectral analysis of $T_n(|f|)^{-1}H_n(f)$, taking into account the following results.
   - By the theory of GLT sequences [13], we have $\{T_n(|f|) - |C_n|\}_n \sim_\sigma 0$, where 0 is the identically zero function. This implies that the singular values of $T_n(|f|) - |C_n|$ are clustered "around" 0.
   - Suppose that f is real, $f \neq 0$ a.e., and f has non-constant sign. By the theory of GLT sequences [13], we have $\{T_n(|f|)^{-1}T_n(f)\}_n \sim_\lambda |f|^{-1}f = \mathrm{sgn}(f)$, which implies that the eigenvalues of $T_n(|f|)^{-1}T_n(f)$ are clustered "around" the two points 1 and -1.
   - Suppose that f is as in the previous item. By the localization result in [28, Theorem 2.1], the eigenvalues of $T_n(|f|)^{-1}T_n(f)$ belong to the open interval $(-1, 1)$.
2. Extend the main results of this paper (Theorems 3.1–3.6) to flipped multilevel (block) Toeplitz matrices, also in view of possible applications to evolutionary partial differential equations in multidimensional domains. This extension presents some challenges. In particular, one of the starting points of the analysis carried out herein is Corollary 2.1, which is deeply connected to [5, Theorem 5]. It would therefore be necessary to first generalize [5, Theorem 5] to the multilevel case. Then, Theorems 3.1–3.3 should be reformulated accordingly, using the multi-index language typical of the multilevel setting and taking into account that f is now a multivariate function. Concerning the generalization to the multilevel case of the localization Theorem 3.5, this might be achieved by exploiting/extending the "multilevel results" in [29, Section 2].

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Data availability

No data was used for the research described in the article.

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