

Are the eigenvalues of preconditioned banded symmetric Toeplitz matrices known in almost closed form?

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Abstract Bogoya, Böttcher, Grudsky, and Maximenko have recently obtained the precise asymptotic expansion for the eigenvalues of a sequence of Toeplitz matrices $\{T_n(f)\}$, under suitable assumptions on the associated generating function f . In this paper, we provide numerical evidence that some of these assumptions can be relaxed and extended to the case of a sequence of preconditioned Toeplitz matrices $\{T_n^{-1}(g)T_n(f)\}$, for f trigonometric polynomial, g nonnegative, not identically zero trigonometric polynomial, $r = f/g$, and where the ratio r plays the same role as f

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in the nonpreconditioned case. Moreover, based on the eigenvalue asymptotics, we devise an extrapolation algorithm for computing the eigenvalues of preconditioned banded symmetric Toeplitz matrices with a high level of accuracy, with a relatively low computational cost, and with potential application to the computation of the spectrum of differential operators.

Keywords (Preconditioned) Toeplitz matrix · Mass and stiffness matrix · Eigenvalues · Eigenvalue asymptotics · Polynomial interpolation · Extrapolation

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1 Introduction

A matrix of size n , having a fixed entry along each diagonal, is called Toeplitz and enjoys the expression

$$[a_{i-j}]_{i,j=1}^n = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & \ddots & \ddots & \ddots & & \vdots \\ a_2 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & a_{-2} \\ \vdots & & \ddots & \ddots & \ddots & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}.$$

Given a complex-valued Lebesgue integrable function $\phi : [-\pi, \pi] \rightarrow \mathbb{C}$, the n th Toeplitz matrix generated by ϕ is defined as

$$T_n(\phi) = [\hat{\phi}_{i-j}]_{i,j=1}^n,$$

where the quantities $\hat{\phi}_k$ are the Fourier coefficients of ϕ , which means

$$\hat{\phi}_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \phi(\theta) e^{-ik\theta} d\theta, \quad k \in \mathbb{Z}.$$

We refer to $\{T_n(\phi)\}_n$ as the Toeplitz sequence generated by ϕ , which in turn is called the generating function of $\{T_n(\phi)\}_n$. In the case where ϕ is real-valued, all the matrices $T_n(\phi)$ are Hermitian and much is known about their spectral properties, from the localization of the eigenvalues to the asymptotic spectral distribution in the Weyl sense: in particular ϕ is the spectral symbol of $\{T_n(\phi)\}_n$, see [7, 14] and the references therein.

More in detail, if ϕ is real-valued and not identically constant, then any eigenvalue of $T_n(\phi)$ belongs to the open set (m_ϕ, M_ϕ) , with m_ϕ, M_ϕ being the essential infimum, the essential supremum of ϕ , respectively. The case of a constant ϕ is trivial: in that case if $\phi = m$ almost everywhere then $T_n(\phi) = m\mathbb{I}_n$ with \mathbb{I}_n denoting the identity of size n . Hence if $M_\phi > 0$ and ϕ is nonnegative almost everywhere, then $T_n(\phi)$ is Hermitian positive definite.

In this paper, we focus our attention on the following setting.

In our context, we say that a function is monotone if it is either increasing or decreasing over the interval $[0, \pi]$.

Under the assumption that $r = f/g$ is monotone, in this paper, we show experimentally that for every integer $\alpha \geq 0$, every n and every $j = 1, \dots, n$, the following asymptotic expansion holds:

$$\lambda_j(\mathcal{P}_n(f, g)) = r(\theta_{j,n}) + \sum_{k=1}^{\alpha} c_k(\theta_{j,n})h^k + E_{j,n,\alpha}, \quad (1)$$

where:

- the eigenvalues of $\mathcal{P}_n(f, g)$ are arranged in nondecreasing or nonincreasing order, depending on whether r is increasing or decreasing;
- $\{c_k\}_{k=1,2,\dots}$ is a sequence of functions from $[0, \pi]$ to \mathbb{R} which depends only on r ;
- $h = \frac{1}{n+1}$ and $\theta_{j,n} = \frac{j\pi}{n+1} = j\pi h$;
- $E_{j,n,\alpha} = O(h^{\alpha+1})$ is the remainder (the error), which satisfies the inequality $|E_{j,n,\alpha}| \leq C_\alpha h^{\alpha+1}$ for some constant C_α depending only on α and r .

In the pure Toeplitz case, that is for $g = 1$ identically, so that $\mathcal{P}_n(f, g) = T_n(f)$ and $r = f$, the result is proven in [4–6], if the RCTP f is monotone and satisfies certain additional assumptions, which include the requirements that $f'(\theta) \neq 0$ for $\theta \in (0, \pi)$ and $f''(\theta) \neq 0$ for $\theta \in \{0, \pi\}$. The symbols

$$f_q(\theta) = (2 - 2 \cos \theta)^q, \quad q = 1, 2, \dots, \quad (2)$$

arise in the discretization of differential equations and are therefore of particular interest. Unfortunately, for these symbols, the requirement that $f''(0) \neq 0$ is not satisfied if $q \geq 2$. In [13], several numerical evidences are reported, showing that the higher order approximation (1) holds even in this “degenerate case.”

Here, as first purpose, we show numerically the same for the preconditioned matrices $\mathcal{P}_n(f, g)$ and, from a theoretical point of view, the numerical testing is complemented by the proof of the above conjecture in the basic case of $\alpha = 0$.

Furthermore, in [13], the authors employed the asymptotic expansion (1) for computing an accurate approximation of $\lambda_j(T_n(f))$ for very large n , provided that the values $\lambda_{j_1}(T_{n_1}(f)), \dots, \lambda_{j_s}(T_{n_s}(f))$ are available for moderate sizes n_1, \dots, n_s with $\theta_{j_1, n_1} = \dots = \theta_{j_s, n_s} = \theta_{j,n}$, $s \geq 2$. The second and main purpose of this paper is to carry out this idea and to support it by numerical experiments, accompanied by an appropriate error analysis in the more general case of the preconditioned matrices $\mathcal{P}_n(f, g)$. In particular, we devise an algorithm to compute $\lambda_j(\mathcal{P}_n(f, g))$ with a high level of accuracy and a relatively low computational cost. The algorithm is completely analogous to the extrapolation procedure, which is employed in the context of Romberg integration (to obtain high precision approximations of an integral from a few coarse trapezoidal approximations [20, Section 3.4], see also [8] for more advanced algorithms). In this regard, the asymptotic expansion (1) plays here the same role as the Euler–Maclaurin summation formula [20, Section 3.3].

The third and last purpose of this paper is to formulate, on the basis of numerical experiments, a conjecture on the higher-order asymptotic of the eigenvalues if the monotonicity assumption on $r = f/g$ is not in force. We also illustrate how this

conjecture can be used along with our extrapolation algorithm in order to compute some of the eigenvalues of $\mathcal{P}_n(f, g)$ in the case where r is nonmonotone.

2 Error bounds for the coefficients c_k in the asymptotic expansion

We start this section by manipulating the error expression implicitly given in (1), the goal being that of using extrapolation methods [8]. In fact, if we assume that the relations in (1) hold, then we can write

$$E_{j,n,0} = \sum_{k=1}^{\alpha} c_k(\theta_{j,n}) h^k + E_{j,n,\alpha}, \tag{3}$$

where $E_{j,n,0} = \lambda_j(\mathcal{P}_n(f, g)) - r(\theta_{j,n})$.

We now suppose to know the eigenvalues for different (small) n_i namely $\{(n_1, \lambda_{j_1}(\mathcal{P}_{n_1}(f, g))), (n_2, \lambda_{j_2}(\mathcal{P}_{n_2}(f, g))), \dots, (n_\alpha, \lambda_{j_\alpha}(\mathcal{P}_{n_\alpha}(f, g)))\}$, where $n_1, n_2, \dots, n_\alpha$ and $j_1, j_2, \dots, j_\alpha$ are chosen in such a way that $j_1/(n_1 + 1) = j_2/(n_2 + 1) = \dots = j_\alpha/(n_\alpha + 1)$.

By defining $h_1 = 1/(n_1 + 1), h_2 = 1/(n_2 + 1), \dots, h_\alpha = 1/(n_\alpha + 1)$, for a given set of eigenvalues, (3) can be written as

$$\begin{aligned} E_{j_1,n_1,0} &= \sum_{k=1}^{\alpha} c_k(\theta_{j_1,n_1}) h_1^k + E_{j_1,n_1,\alpha}, \\ E_{j_2,n_2,0} &= \sum_{k=1}^{\alpha} c_k(\theta_{j_2,n_2}) h_2^k + E_{j_2,n_2,\alpha}, \\ E_{j_3,n_3,0} &= \sum_{k=1}^{\alpha} c_k(\theta_{j_3,n_3}) h_3^k + E_{j_3,n_3,\alpha}, \\ &\vdots \\ E_{j_\alpha,n_\alpha,0} &= \sum_{k=1}^{\alpha} c_k(\theta_{j_\alpha,n_\alpha}) h_\alpha^k + E_{j_\alpha,n_\alpha,\alpha}. \end{aligned} \tag{4}$$

Let c, \tilde{c} be the vectors

$$c = [c_1, c_2, \dots, c_\alpha]^T; \quad \tilde{c} = [\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_\alpha]^T,$$

and let A be the coefficient matrix of size $\alpha \times \alpha$ with $(A)_{i,j} = h_i^j$. Hence, the set of (4) can be written in matrix form as

$$Ac = b_0 - b_\alpha, \tag{5}$$

where $b_0 = [E_{j_1,n_1,0}, E_{j_2,n_2,0}, \dots, E_{j_\alpha,n_\alpha,0}]^T$ and $b_\alpha = [E_{j_1,n_1,\alpha}, E_{j_2,n_2,\alpha}, \dots, E_{j_\alpha,n_\alpha,\alpha}]^T$. Furthermore, by neglecting the higher order errors, we may define an approximation \tilde{c} of c according to the expression below

$$A\tilde{c} = b_0. \tag{6}$$

By solving the linear system of equations above, the approximation of c is easily obtained since the matrix size is very small. In a subsequent step, we derive upper-bounds for $|\tilde{c} - c|$: in reality, (5) and (6) leads to

$$A(\tilde{c} - c) = b_\alpha. \tag{7}$$

If we define $\Delta c = \tilde{c} - c$ and $\eta_i = \frac{E_{j_i, n_i, \alpha}}{h_i^{\alpha+1}}$ for $i = 1, \dots, \alpha$, then the system (7) can be written as

$$A\Delta c = \begin{bmatrix} \eta_1 h_1^{\alpha+1} \\ \eta_2 h_2^{\alpha+1} \\ \vdots \\ \eta_\alpha h_\alpha^{\alpha+1} \end{bmatrix}, \tag{8}$$

with $|\eta_i| \leq C_\alpha$ for $i = 1, \dots, \alpha$, where C_α is a constant. The coefficient matrix can be expressed as

$$A = \begin{bmatrix} h_1 & h_1^2 & \dots & h_1^\alpha \\ h_2 & h_2^2 & \dots & h_2^\alpha \\ \vdots & \vdots & \ddots & \vdots \\ h_\alpha & h_\alpha^2 & \dots & h_\alpha^\alpha \end{bmatrix} = \begin{bmatrix} h_1 & & & \\ & h_2 & & \\ & & \ddots & \\ & & & h_\alpha \end{bmatrix} V(h_1, \dots, h_\alpha),$$

where $V(h_1, \dots, h_\alpha)$ is the Vandermonde matrix of order α corresponding to h_1, \dots, h_α .

By assuming $W = V^{-1}(h_1, \dots, h_\alpha)$, we deduce

$$(W)_{i,j} = \begin{cases} (-1)^{\alpha-i} \left(\frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} h_{k_1} \dots h_{k_{\alpha-i}}}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} (h_j - h_k)} \right) & 1 \leq i < \alpha, \\ \frac{1}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} (h_j - h_k)} & i = \alpha. \end{cases} \tag{9}$$

Therefore for the inversion of the matrix A , we have

$$(A^{-1})_{i,j} = \begin{cases} (-1)^{\alpha-i} \left(\frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} h_{k_1} \dots h_{k_{\alpha-i}}}{h_j \prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} (h_j - h_k)} \right) & 1 \leq i < \alpha, \\ \frac{1}{h_j \prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} (h_j - h_k)} & i = \alpha, \end{cases} \tag{10}$$

and we can obtain an explicit expression for $(\Delta c)_i, i = 1, \dots, \alpha$, that is

$$(\Delta c)_i = \sum_{j=1}^{\alpha} (A^{-1})_{i,j} \eta_j h_j^{\alpha+1}. \tag{11}$$

Case 1 If $i = \alpha$, then

$$(\Delta c)_\alpha = \sum_{j=1}^{\alpha} \frac{\eta_j h_j^{\alpha+1}}{h_j \prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} (h_j - h_k)}.$$

Whence, from the fact that $|\eta_i| \leq C_\alpha$ for $i = 1, \dots, \alpha$,

$$|(\Delta c)_\alpha| \leq \sum_{j=1}^{\alpha} \frac{|\eta_j| h_j^{\alpha+1}}{h_j \prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} |h_j - h_k|} \leq \sum_{j=1}^{\alpha} \frac{C_\alpha h_j^\alpha}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} |h_j - h_k|}.$$

With the choice $h_j = \frac{1}{m^{j-1}} h_1$ for $j = 1, \dots, \alpha$, m positive integer, we have

$$\begin{aligned} |(\Delta c)_\alpha| &\leq C_\alpha \sum_{j=1}^{\alpha} \frac{\left(\frac{h_1}{m^{j-1}}\right)^\alpha}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} h_1 \left| \frac{1}{m^{j-1}} - \frac{1}{m^{k-1}} \right|} = C_\alpha h_1^\alpha \sum_{j=1}^{\alpha} \frac{\left(\frac{1}{m^{j-1}}\right)^\alpha}{h_1^{\alpha-1} \prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} \left| \frac{1}{m^{j-1}} - \frac{1}{m^{k-1}} \right|} \\ &= h_1 C_\alpha \sum_{j=1}^{\alpha} \frac{\left(\frac{1}{m^{j-1}}\right)^\alpha}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} \left| \frac{1}{m^{j-1}} - \frac{1}{m^{k-1}} \right|} = O(h_1). \end{aligned}$$

Case 2 If $i = 1, \dots, \alpha - 1$, then

$$(\Delta c)_i = \sum_{j=1}^{\alpha} (-1)^{\alpha-i} \eta_j h_j^{\alpha+1} \frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} h_{k_1} \cdots h_{k_{\alpha-i}}}{h_j \prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} (h_j - h_k)},$$

that is different from the case $i = \alpha$ just for the numerator

$$\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} h_{k_1} \cdots h_{k_{\alpha-i}}.$$

As a consequence,

$$|(\Delta c)_i| \leq C_\alpha \sum_{j=1}^{\alpha} h_j^\alpha \frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} h_{k_1} \cdots h_{k_{\alpha-i}}}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} |h_j - h_k|}.$$

With the choice $h_j = \frac{1}{m^{j-1}}h_1$ for $j = 1, \dots, \alpha$, we infer

$$\begin{aligned}
 |(\Delta c)_i| &\leq C_\alpha \sum_{j=1}^\alpha \left(\frac{h_1}{m^{j-1}}\right)^\alpha \frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} h_1^{\alpha-i} \left(\frac{1}{m^{k_1-1}} \frac{1}{m^{k_2-1}} \dots \frac{1}{m^{k_{\alpha-i}-1}}\right)}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} h_1 \left|\frac{1}{m^{j-1}} - \frac{1}{m^{k-1}}\right|} \\
 &= C_\alpha \sum_{j=1}^\alpha \left(\frac{1}{m^{j-1}}\right)^\alpha \left(\frac{h_1^\alpha h_1^{\alpha-i}}{h_1^{\alpha-1}}\right) \frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} \left(\frac{1}{m^{k_1-1}} \frac{1}{m^{k_2-1}} \dots \frac{1}{m^{k_{\alpha-i}-1}}\right)}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} \left|\frac{1}{m^{j-1}} - \frac{1}{m^{k-1}}\right|} \\
 &= h_1^{\alpha-i+1} C_\alpha \sum_{j=1}^\alpha \left(\frac{1}{m^{j-1}}\right)^\alpha \frac{\sum_{\substack{1 \leq k_1 < \dots < k_{\alpha-i} \leq \alpha \\ k_1, \dots, k_{\alpha-i} \neq j}} \left(\frac{1}{m^{k_1-1}} \frac{1}{m^{k_2-1}} \dots \frac{1}{m^{k_{\alpha-i}-1}}\right)}{\prod_{\substack{1 \leq k \leq \alpha \\ k \neq j}} \left|\frac{1}{m^{j-1}} - \frac{1}{m^{k-1}}\right|} = O(h_1^{\alpha-i+1}).
 \end{aligned}$$

As a conclusion, with the choice $h_j = \frac{1}{m^{j-1}}h_1$ for $j = 1, \dots, \alpha$ and under the assumption that the asymptotic expansion reported in (1) is true, we deduce

$$|(\Delta c)_i| = O(h_1^{\alpha-i+1}), \tag{12}$$

for $i = 1, \dots, \alpha$.

3 Error bounds for numerically approximated eigenvalues

The goal of this short section is to provide error bounds based on the linear system in (6) for the computation of the eigenvalues of $\mathcal{P}_n(f, g)$: of course, these error bounds are based on the conjecture that the relations reported in (1) are true. However, as we can see in Section 4, the numerical tests fully support the existence of the considered asymptotic expansion.

Indeed, as already observed, by solving (6), we can approximate c_k . Once we have the values of c_k , we can approximate the eigenvalues λ_{j_β} of a large dimension matrix of size n_β , here $n_\beta + 1 = m^{\beta-1}(n_1 + 1)$. The asymptotic expansion (3) can be written as

$$E_{j_\beta, n_\beta, 0} = \bar{h}_\beta^T c + E_{j_\beta, n_\beta, \alpha}. \tag{13}$$

By subtraction $\bar{h}_\beta^T \tilde{c}$ from both sides of the equation above, we find

$$\begin{aligned}
 E_{j_\beta, n_\beta, 0} - \bar{h}_\beta^T \tilde{c} &= \bar{h}_\beta^T (c - \tilde{c}) + E_{j_\beta, n_\beta, \alpha}, \\
 \lambda_j(\mathcal{P}_{n_\beta}(f, g)) - r(\theta_{j, n_\beta}) - \bar{h}_\beta^T \tilde{c} &= \bar{h}_\beta^T \Delta c + E_{j_\beta, n_\beta, \alpha}, \\
 \left| \lambda_j(\mathcal{P}_{n_\beta}(f, g)) - r(\theta_{j, n_\beta}) - \bar{h}_\beta^T \tilde{c} \right| &\leq \sum_{i=1}^\alpha h_\beta^i |(\Delta c)_i| + |E_{j_\beta, n_\beta, \alpha}|, \\
 \left| \lambda_j(\mathcal{P}_{n_\beta}(f, g)) - r(\theta_{j, n_\beta}) - \bar{h}_\beta^T \tilde{c} \right| &\leq \sum_{i=1}^\alpha h_\beta^i |(\Delta c)_i| + C_\alpha h_\beta^{\alpha+1},
 \end{aligned} \tag{14}$$

where $\bar{h}_\beta = [h_\beta, h_\beta^2, \dots, h_\beta^{\alpha+1}]^T$, $|E_{j_\beta, n_\beta, \alpha}| \leq C_\alpha h_\beta^{\alpha+1}$ for some constant C_α and $|(\Delta c)_i|$ is given in (12).

4 Numerical tests

In this section, we want to present a few numerical experiments to support the asymptotic expansion (1) in the case where one or more properties of the following list are satisfied:

1. $f''(0) \neq 0$ (see Example 1, Example 3, and Example 5),
2. $f''(0) = 0$ (see Example 2 and Example 4),
3. $\min g > 0$ (see Example 1, Example 2, and Example 5),
4. $\min g = 0$ (see Example 3 and Example 4),
5. $r = f/g$ is non monotone (see Example 5).

The approximation of eigenvalues of large matrices in each case is also computed. The expansion (1) for $\alpha = 4$ is

$$\begin{aligned} \lambda_j(\mathcal{P}_n(f, g)) &= r(\theta_{j,n}) + c_1(\theta_{j,n})h + c_2(\theta_{j,n})h^2 + c_3(\theta_{j,n})h^3 + c_4(\theta_{j,n})h^4 + E_{j,n,4}, \\ E_{j,n,0} &= \lambda_j(\mathcal{P}_n(f, g)) - r(\theta_{j,n}) = c_1(\theta_{j,n})h + c_2(\theta_{j,n})h^2 + c_3(\theta_{j,n})h^3 \\ &\quad + c_4(\theta_{j,n})h^4 + E_{j,n,4}. \end{aligned} \tag{15}$$

In all numerical examples, we choose four matrix-size values, that is n_i for $i \in \{1, 2, 3, 4\}$, in a way that they satisfy $n_i = m^{i-1}(n_1 + 1) - 1$, with m being a positive integer. The expansion (15) for the set of the four dimensions n_i can be written as

$$\begin{aligned} E_{j_1, n_1, 0} &= c_1(\theta_{j_1, n_1})h_1 + c_2(\theta_{j_1, n_1})h_1^2 + c_3(\theta_{j_1, n_1})h_1^3 + c_4(\theta_{j_1, n_1})h_1^4 + E_{j_1, n_1, 4}, \\ E_{j_2, n_2, 0} &= c_1(\theta_{j_2, n_2})h_2 + c_2(\theta_{j_2, n_2})h_2^2 + c_3(\theta_{j_2, n_2})h_2^3 + c_4(\theta_{j_2, n_2})h_2^4 + E_{j_2, n_2, 4}, \\ E_{j_3, n_3, 0} &= c_1(\theta_{j_3, n_3})h_3 + c_2(\theta_{j_3, n_3})h_3^2 + c_3(\theta_{j_3, n_3})h_3^3 + c_4(\theta_{j_3, n_3})h_3^4 + E_{j_3, n_3, 4}, \\ E_{j_4, n_4, 0} &= c_1(\theta_{j_4, n_4})h_4 + c_2(\theta_{j_4, n_4})h_4^2 + c_3(\theta_{j_4, n_4})h_4^3 + c_4(\theta_{j_4, n_4})h_4^4 + E_{j_4, n_4, 4}, \end{aligned} \tag{16}$$

where $h_i = \frac{1}{n_i+1}$ and $j_i = m^{i-1}j_1$ for $i \in \{1, 2, 3, 4\}$. Notice that $\theta_{j_i, n_i} = \theta_{j_1, n_1} = \bar{\theta}$ for a fixed $j_1 \in \{1, 2, \dots, n_1\}$. We are interested in the numerical approximation of $c_i(\bar{\theta})$ for $i \in \{1, 2, 3, 4\}$ and then in the precise numerical approximation of the eigenvalue of $\mathcal{P}_n(f, g)$ for large n . The set of (16) can be written as

$$\begin{aligned} E_{j_1, n_1, 0} &= \tilde{c}_1(\bar{\theta})h_1 + \tilde{c}_2(\bar{\theta})h_1^2 + \tilde{c}_3(\bar{\theta})h_1^3 + \tilde{c}_4(\bar{\theta})h_1^4, \\ E_{j_2, n_2, 0} &= \tilde{c}_1(\bar{\theta})h_2 + \tilde{c}_2(\bar{\theta})h_2^2 + \tilde{c}_3(\bar{\theta})h_2^3 + \tilde{c}_4(\bar{\theta})h_2^4, \\ E_{j_3, n_3, 0} &= \tilde{c}_1(\bar{\theta})h_3 + \tilde{c}_2(\bar{\theta})h_3^2 + \tilde{c}_3(\bar{\theta})h_3^3 + \tilde{c}_4(\bar{\theta})h_3^4, \\ E_{j_4, n_4, 0} &= \tilde{c}_1(\bar{\theta})h_4 + \tilde{c}_2(\bar{\theta})h_4^2 + \tilde{c}_3(\bar{\theta})h_4^3 + \tilde{c}_4(\bar{\theta})h_4^4. \end{aligned} \tag{17}$$

We solve the system of linear equations above for $j_1 \in \{1, 2, \dots, n_1\}$ to compute $\tilde{c}_i(\bar{\theta})$. The computed \tilde{c}_i are used to approximate the eigenvalues of large size n_β by exploiting the following relation

$$\tilde{\lambda}_{j_\beta}(\mathcal{P}_{n_\beta}(f, g)) = r(\theta_{j_\beta, n_\beta}) + \bar{h}_\beta^T \tilde{c}. \tag{18}$$

Example 1 Let g, f , and r be the functions defined as

$$\begin{aligned}
 f(\theta) &= 4 - 2 \cos(\theta) - 2 \cos(2\theta) = (2 - 2 \cos(\theta))(3 + 2 \cos(\theta)), \\
 g(\theta) &= 3 + 2 \cos(\theta), \\
 r(\theta) &= \frac{f(\theta)}{g(\theta)} = 2 - 2 \cos(\theta),
 \end{aligned}$$

where $\theta \in [0, \pi]$. The graphs of generating functions are shown in left panel of Fig. 1, and the approximations \tilde{c}_k , for $k = 1, 2, 3, 4$ are shown in the right panel. Note that $g(\theta) > 0, \forall \theta \in [0, \pi], f''(0) \neq 0$, and furthermore r is monotone. We set $n = n_1 \in \{40, 60, 80, 100\}$ and $m = 2$.

Example 2 Let g, f , and r be the functions defined as

$$\begin{aligned}
 f(\theta) &= 20 - 30 \cos(\theta) + 12 \cos(2\theta) - 2 \cos(3\theta) = (2 - 2 \cos(\theta))^3, \\
 g(\theta) &= 3 + 2 \cos(\theta), \\
 r(\theta) &= \frac{f(\theta)}{g(\theta)} = \frac{(2 - 2 \cos(\theta))^3}{3 + 2 \cos(\theta)},
 \end{aligned}$$

where $\theta \in [0, \pi]$. The graphs of generating functions are shown in left panel of Fig. 2, and the approximations \tilde{c}_k , for $k = 1, 2, 3, 4$ are shown in the right panel. Remark that $g(\theta) > 0, \forall \theta \in [0, \pi], f''(0) = 0$, and furthermore r is monotone. We set $n = n_1 \in \{40, 60, 80, 100\}$ and $m = 2$.

There is an important issue to discuss here. Both the functions f and r attain the minimum at $\theta = 0$ with a very high order. Indeed, we have $f(\theta), r(\theta) \approx \theta^6$, with $\phi_1 \approx \phi_2$ being the symmetric, transitive relation telling that there exist positive constants $c, C > 0$ such that $c\phi_1 \leq \phi_2 \leq C\phi_1$ on the whole definition domain $[0, \pi]$. Therefore for fixed j (independent of n) the j th smallest eigenvalue of $\mathcal{P}_n(f, g)$ is asymptotic to $k_j h^6, k_j$ positive constant depending on j but not on n : the reader is referred to [17] for the preconditioned case with the limitation $j = 1$ and to [1] and references therein for very elegant and precise estimates regarding the pure Toeplitz case.

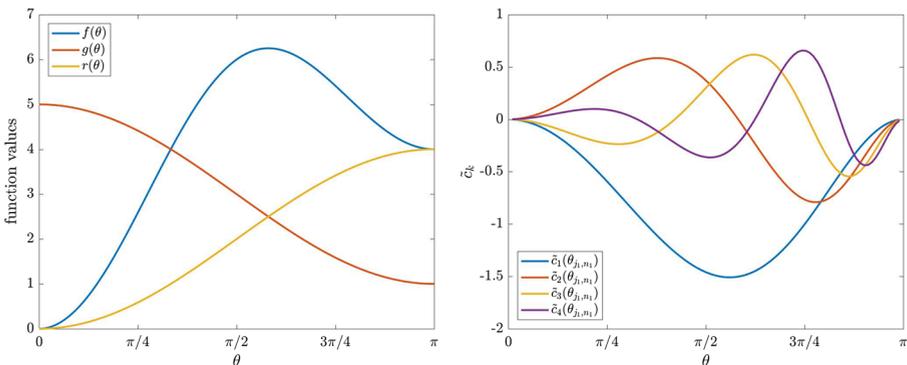


Fig. 1 Example 1: Generating functions (f, g , and r) and \tilde{c}_k for $k = 1, 2, 3, 4$

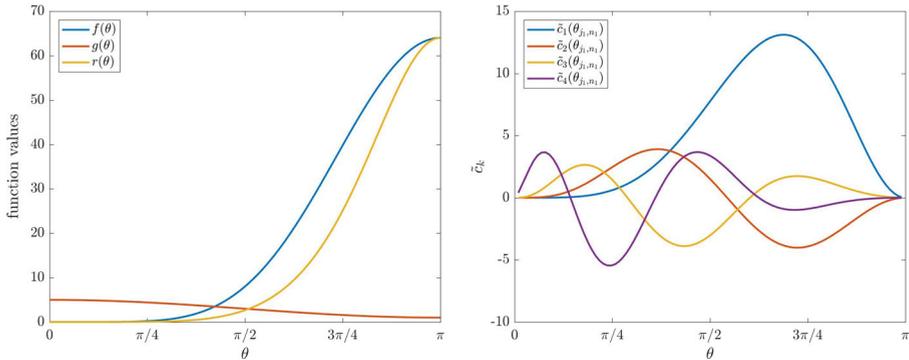


Fig. 2 Example 2: generating functions (f , g , and r) and \tilde{c}_k for $k = 1, 2, 3, 4$

Now if we fix j and we put together $\lambda_j(\mathcal{P}_n(f, g)) \approx h^6$ with relations (3)–(4), then the only possibility for avoiding a contradiction is that the functions $c_1(\theta), c_2(\theta), c_3(\theta), c_4(\theta), c_5(\theta)$ all vanish at $\theta = 0$.

The approximations \tilde{c}_k , for $k = 1, 2, 3, 4$ shown in the right panel of Fig. 2 are coherent with the above mathematical conclusion and in fact all these approximations vanish simultaneously at $\theta = 0$ (the fifth is not displayed, but we computed it and it also equals to zero at $\theta = 0$, while, as expected from an extension of the results by [1] to the preconditioned Toeplitz case, the sixth is nonzero at $\theta = 0$).

Since the argument and the conclusions are the very same, we anticipate that the discussion can be repeated verbatim for Example 4, where the functions f and r attain the minimum at $\theta = 0$ with order 10. As a consequence, we expect that the functions $c_1(\theta), \dots, c_9(\theta)$ all simultaneously vanish at $\theta = 0$, while $c_{10}(0) \neq 0$: this is confirmed for the first four of them as reported in the right panel of Fig. 4.

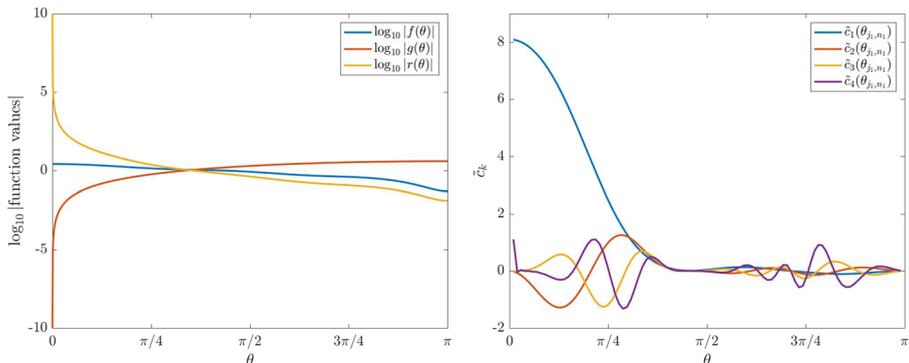


Fig. 3 Example 3: generating functions (f , g , and r) and \tilde{c}_k for $k = 1, 2, 3, 4$

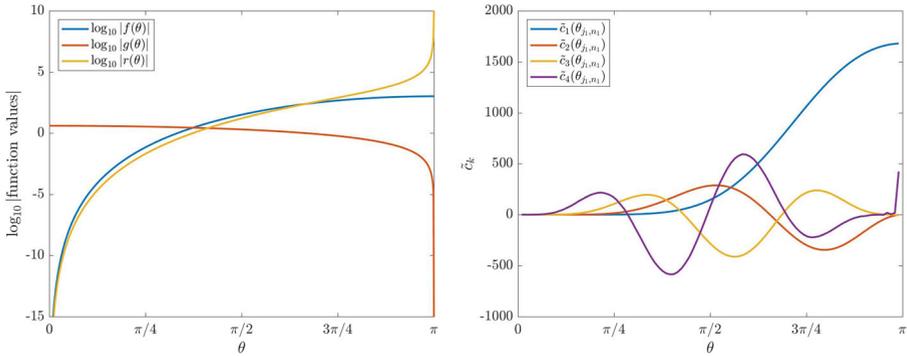


Fig. 4 Example 4: generating functions (f , g , and r) and \tilde{c}_k for $k = 1, 2, 3, 4$

Example 3 Let g , f , and r be the functions defined as

$$f(\theta) = 1 + \cos(\theta) + \frac{1}{4} \cos(2\theta) + \frac{1}{5} \cos(3\theta) + \frac{1}{10} \cos(4\theta) + \frac{1}{10} \cos(5\theta),$$

$$g(\theta) = 2 - 2 \cos(\theta),$$

$$r(\theta) = \frac{f(\theta)}{g(\theta)} = \frac{1 + \cos(\theta) + \frac{1}{4} \cos(2\theta) + \frac{1}{5} \cos(3\theta) + \frac{1}{10} \cos(4\theta) + \frac{1}{10} \cos(5\theta)}{2 - 2 \cos(\theta)},$$

where $\theta \in [0, \pi]$. The graphs of generating functions are shown in left panel of Fig. 3, and the approximations \tilde{c}_k , for $k = 1, 2, 3, 4$ are shown in the right panel. Note that $\min g(\theta) = 0, \forall \theta \in [0, \pi], f''(0) \neq 0$, and furthermore r is monotone. We set $n = n_1 \in \{40, 60, 80, 100\}$ and $m = 2$.

Example 4 Let g , f , and r be the functions defined as

$$f(\theta) = 252 - 420 \cos(\theta) + 240 \cos(2\theta) - 90 \cos(3\theta) + 20 \cos(4\theta) - 2 \cos(5\theta) = (2 - 2 \cos(\theta))^5,$$

$$g(\theta) = 2 + 2 \cos(\theta),$$

$$r(\theta) = \frac{f(\theta)}{g(\theta)} = \frac{(2 - 2 \cos(\theta))^5}{2 + 2 \cos(\theta)},$$

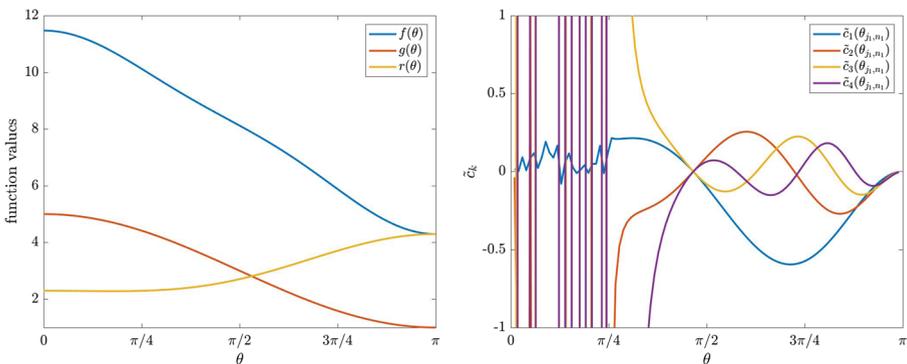


Fig. 5 Example 5: generating functions (f , g , and r) and \tilde{c}_k for $k = 1, 2, 3, 4$

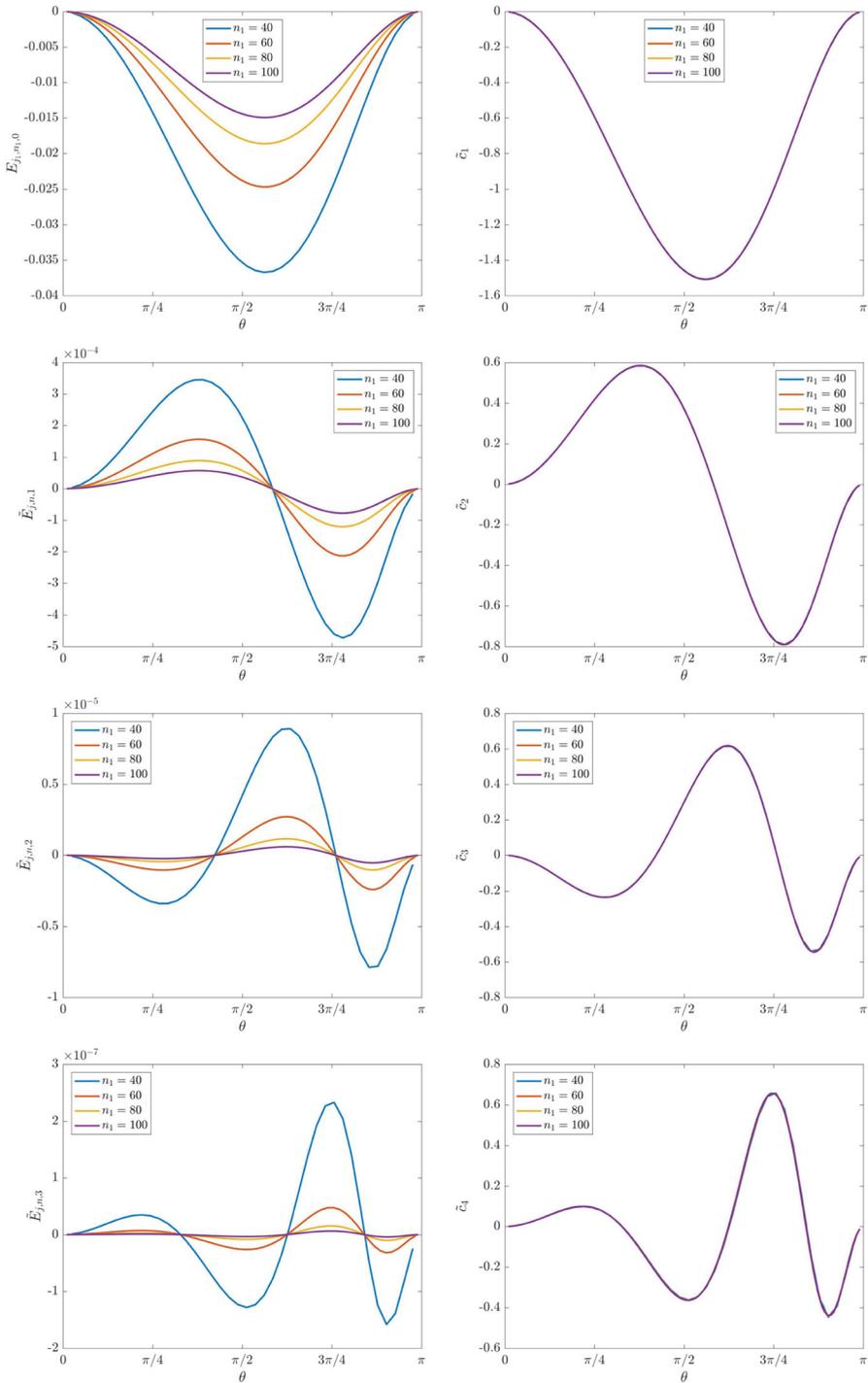


Fig. 6 Example 1: $E_{j,n,0}$, $\tilde{E}_{j,n,k}$ ($k = 1, 2, 3$), and \tilde{c}_k ($k = 1, 2, 3, 4$), for $n = n_1 = \{40, 60, 80, 100\}$

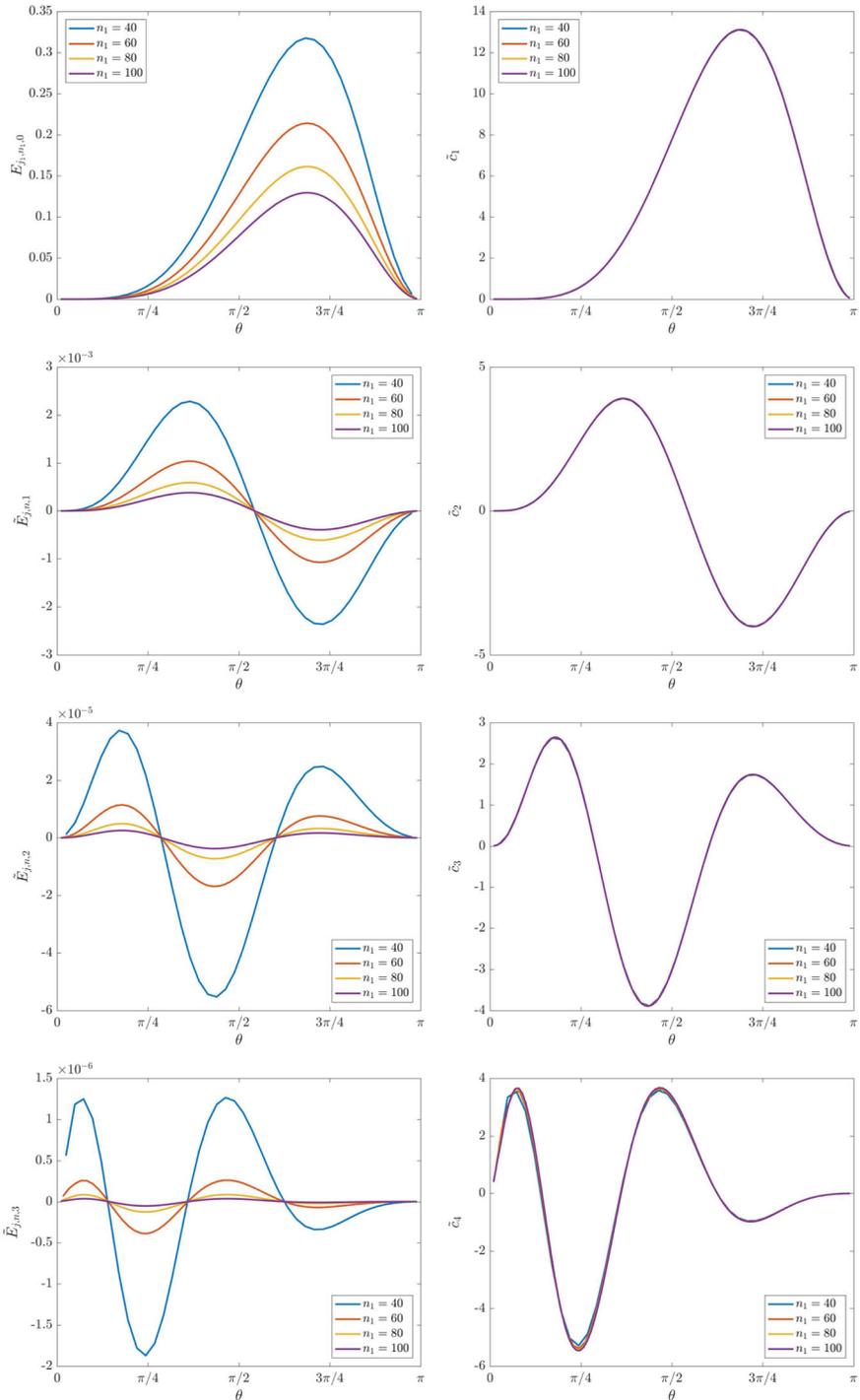


Fig. 7 Example 2: $E_{j,n,0}$, $\tilde{E}_{j,n,k}$ ($k = 1, 2, 3$), and \tilde{c}_k ($k = 1, 2, 3, 4$), for $n = n_1 = \{40, 60, 80, 100\}$

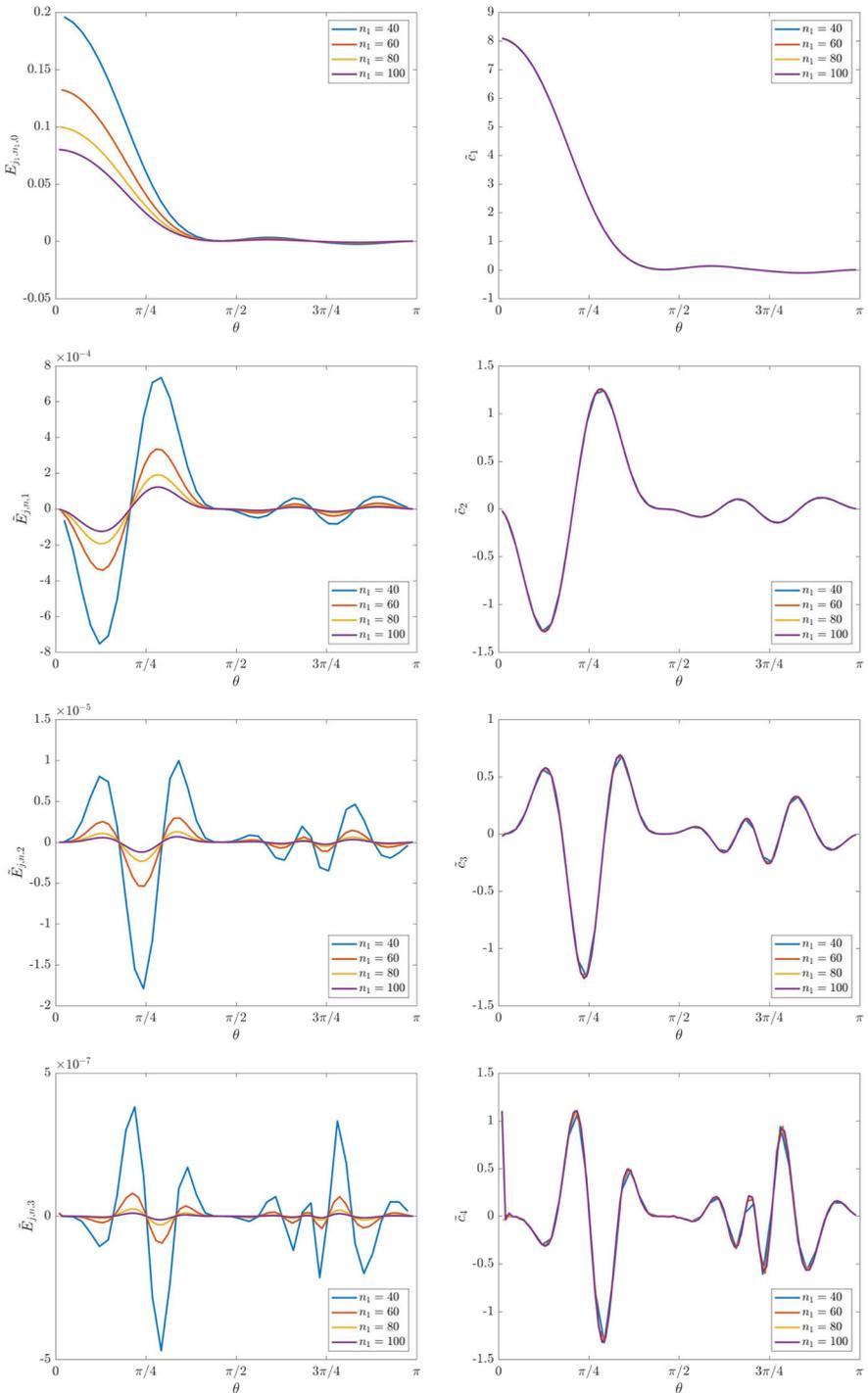


Fig. 8 Example 3: $E_{j,n,k}$, $\tilde{E}_{j,n,k}$ ($k = 1, 2, 3$), and \tilde{c}_k ($k = 1, 2, 3, 4$), for $n = n_1 = \{40, 60, 80, 100\}$

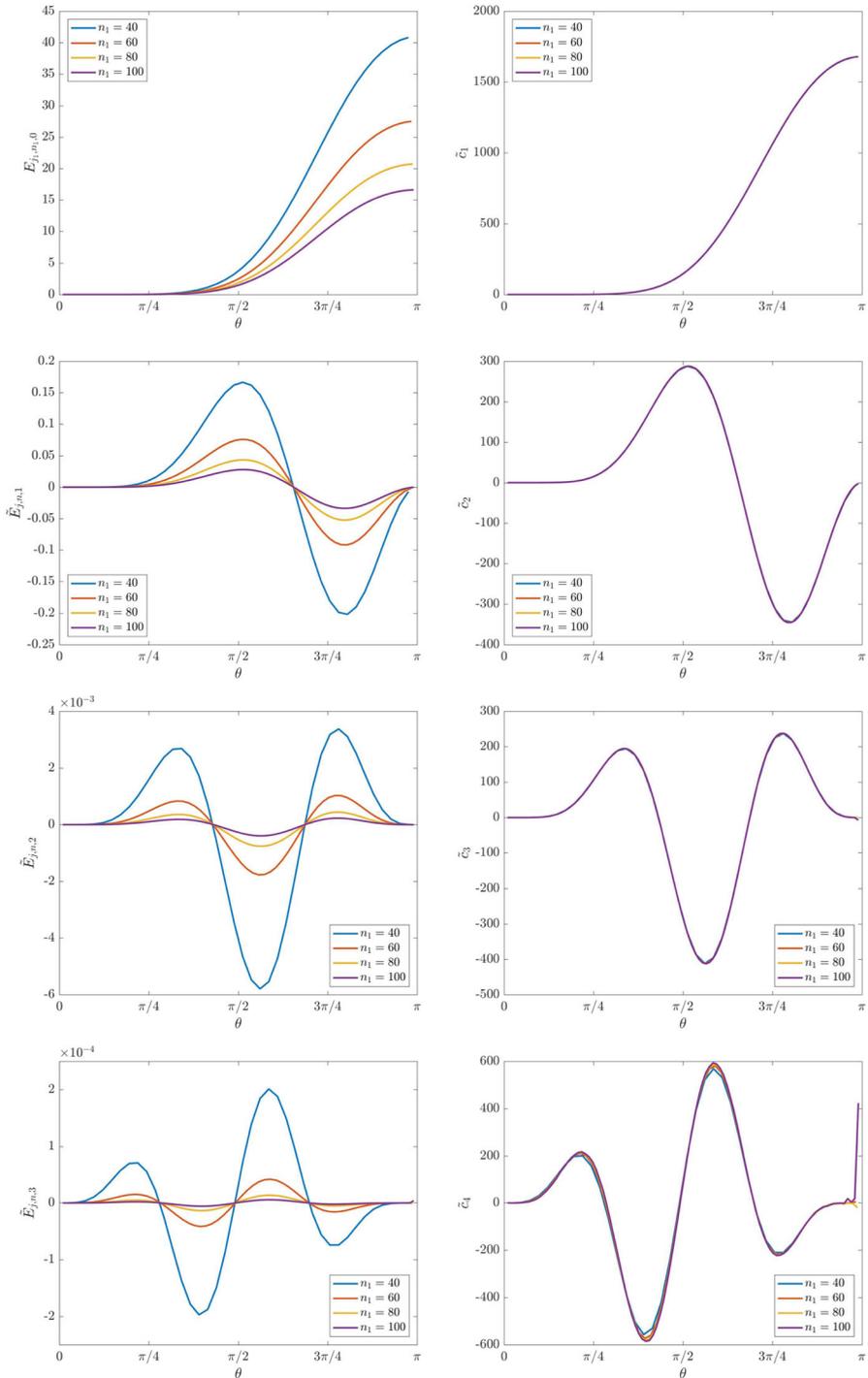


Fig. 9 Example 4: $E_{j,n,0}$, $\tilde{E}_{j,n,k}$ ($k = 1, 2, 3$), and \tilde{c}_k ($k = 1, 2, 3, 4$), for $n = n_1 = \{40, 60, 80, 100\}$

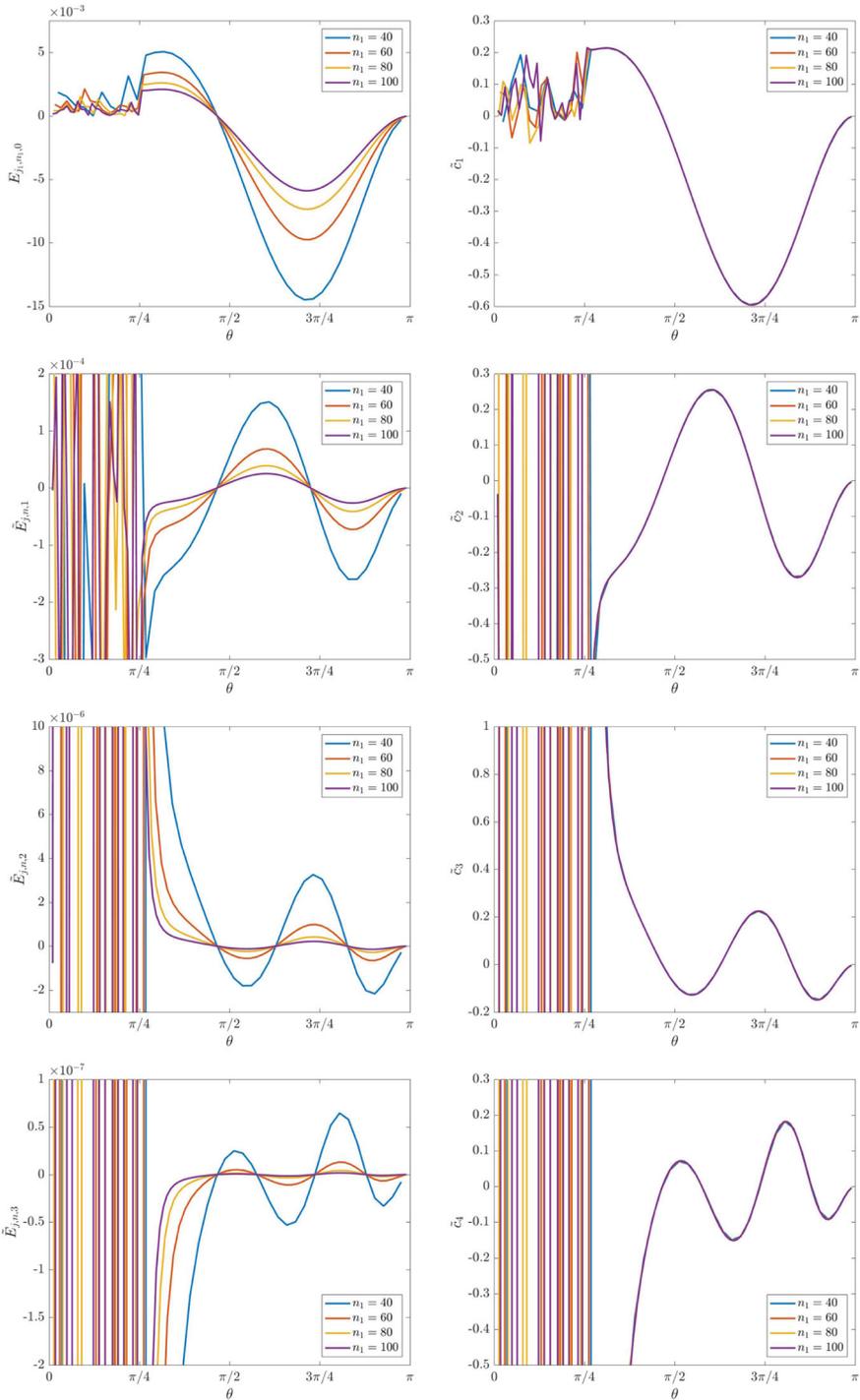


Fig. 10 Example 5: $E_{j,n,0}$, $\tilde{E}_{j,n,k}$ ($k = 1, 2, 3$), and \tilde{c}_k ($k = 1, 2, 3, 4$), for $n = n_1 = \{40, 60, 80, 100\}$

where $\theta \in [0, \pi]$. The graphs of generating functions are shown in left panel of Fig. 4, and the approximations \tilde{c}_k , for $k = 1, 2, 3, 4$ are shown in the right panel. Remark that $\min g(\theta) = 0, \forall \theta \in [0, \pi], f''(0) = 0$, and furthermore r is monotone. We set $n = n_1 \in \{40, 60, 80, 100\}$ and $m = 2$.

Example 5 Let g, f , and r be the functions defined as

$$f(\theta) = \frac{136}{17} + \frac{56}{17} \cos(\theta) - \frac{2}{17} \cos(2\theta) + \frac{5}{17} \cos(3\theta) = (3 - \cos(\theta) + \frac{5}{17} \cos(2\theta))(3 + 2 \cos(\theta)),$$

$$g(\theta) = 3 + 2 \cos(\theta),$$

$$r(\theta) = \frac{f(\theta)}{g(\theta)} = 3 - \cos(\theta) + \frac{5}{17} \cos(2\theta),$$

where $\theta \in [0, \pi]$. The graphs of generating functions are shown in left panel of Fig. 5, and the approximations \tilde{c}_k , for $k = 1, 2, 3, 4$ are shown in the right panel. Notice that $\min g(\theta) > 0, \forall \theta \in [0, \pi], f''(0) \neq 0$, and furthermore r is non monotone. We set $n = n_1 \in \{40, 60, 80, 100\}$ and $m = 2$.

The numerical tests related to Examples 1 and 2, as in Figs. 6 and 7, show that the error expansion (1) behaves as expected. In Fig. 11, we also see that the approximated \tilde{c}_k can be used for a large n to approximate the error term to (or almost to) machine precision.

In the numerical tests associated with Examples 3 and 4, as in Figs. 8 and 9, we observe again that the error expansion is in accordance with (1). We also note a slight deviation for the largest eigenvalue and this has to be expected since we have $r(\theta_{1,n}) \rightarrow \infty$ as $n \rightarrow \infty$ for Example 3 (on the other hand for Example 4 we notice $r(\theta_{n,n}) \rightarrow \infty$ as $n \rightarrow \infty$). However, the approximation of the eigenvalues of $\mathcal{P}_n(f, g)$ is excellent and almost to machine precision as reported in Fig. 12.

In the numerical test related to Example 5, we have a non monotone region for $\theta \in [0, 2 \tan^{-1}(\sqrt{3/17})]$ where the proposed expansion does not work. Indeed, additional errors are introduced when compared to $E_{j,n,0}$, since the sampling of $r(\theta_{j_1, n_1})$ leads to a poorer approximation after ordering than the procedure given by sampling $r(\theta_{j, n_7})$ first and then picking samples after ordering. However, the expansion is

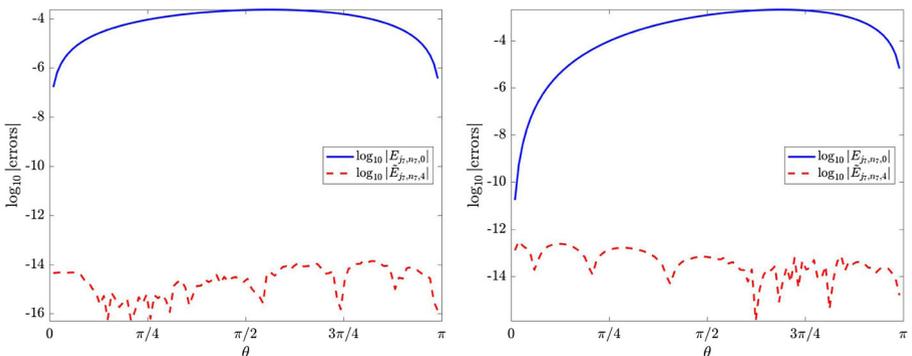


Fig. 11 Example 1 and 2: the errors $\log_{10} |E_{j_7, n_7, 0}|$ and $\log_{10} |\tilde{E}_{j_7, n_7, 4}|$ for the 100 indices j_7 of $n_7 = 6463$ in (18), corresponding to $n_1 = 100$, and using $\tilde{c}_k, k = 1, 2, 3, 4$, computed with $m = 2$

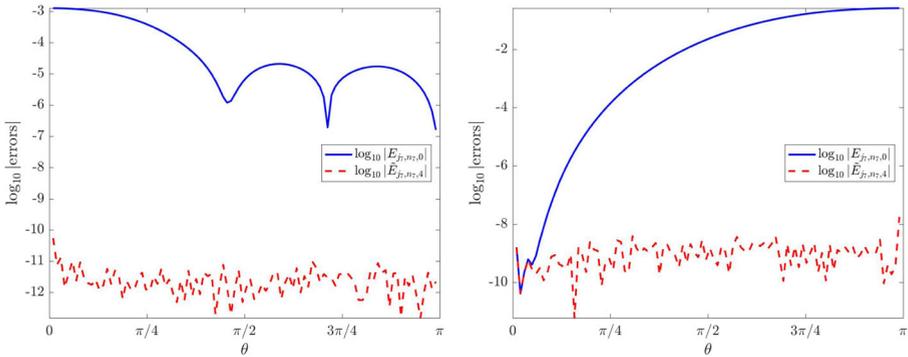


Fig. 12 Example 3 and 4: the errors $\log_{10} |E_{j_7, n_7, 0}|$ and $\log_{10} |\tilde{E}_{j_7, n_7, 4}|$ for the 100 indices j_7 of $n_7 = 6463$ in (18), corresponding to $n_1 = 100$, and using $\tilde{c}_k, k = 1, 2, 3, 4$, computed with $m = 2$

confirmed for the rest of the domain, as seen in Fig. 10. Furthermore, in Fig. 13, the expansion works well again for the monotone part, by allowing an approximation almost to machine precision of the eigenvalues of $\mathcal{P}_n(f, g)$.

However, even if the eigenvalues lying in the non monotone region give raise to an irregular error pattern, it seems that there exists a kind of ‘deformed’ periodicity in the error, like it is formally proven, without deformations, for the eigenvalues of $T_n(f), f(\theta) = 2 - 2 \cos(\omega\theta), \omega \geq 2$ integer, and $g(\theta) = 1$ (see [12]). The latter observation indicates that a more complete study of this ‘deformed’ periodicity has to be considered in the future.

We finally observe that remarkable numerical results for the eigenvalues of $\mathcal{P}_n(f, g)$, as reported in Figs. 11, 12 and 13, really answer in the positive to the question posed in the title of the paper. In fact, we obtain almost machine precision for

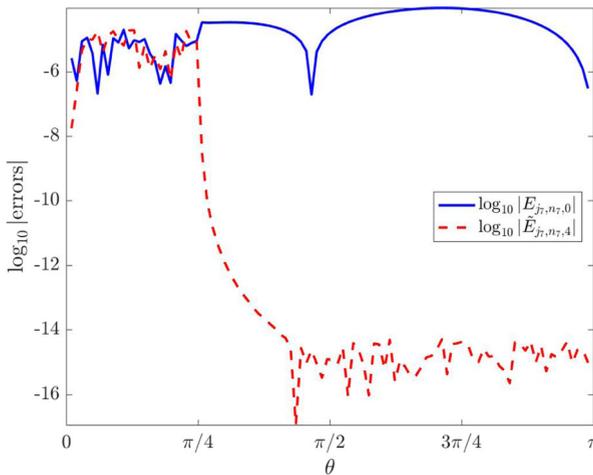


Fig. 13 Example 5: the errors $\log_{10} |E_{j_7, n_7, 0}|$ and $\log_{10} |\tilde{E}_{j_7, n_7, 4}|$ for the 100 indices j_7 of $n_7 = 6463$ in (18), corresponding to $n_1 = 100$, and using $\tilde{c}_k, k = 1, 2, 3, 4$, computed with $m = 2$. Note the non monotone part, $\theta \in [0, 2 \tan^{-1}(\sqrt{3/17})]$, where the error is not improved

the computation of the spectrum of $\mathcal{P}_n(f, g)$, for large n and only working with few really small matrices.

5 Conclusions

Bogoya et al. [4–6] have recently obtained the precise asymptotic expansion for the eigenvalues of a sequence of Toeplitz matrices $\{T_n(f)\}$, under suitable assumptions on the associated generating function f . In this paper, we have shown numerical evidence that some of these assumptions can be relaxed and extended to the case of a sequence of preconditioned Toeplitz matrices $\{\mathcal{P}_n(f, g) = T_n^{-1}(g)T_n(f)\}$, for f trigonometric polynomial, g nonnegative, not identically zero trigonometric polynomial, $r = f/g$, and where the ratio r plays the same role as f in the non-preconditioned case. The first-order asymptotic term of the expansion has been also proven using purely linear algebra tools.

Moreover, based on the eigenvalue asymptotics, we devised an extrapolation algorithm for computing the eigenvalues of preconditioned banded symmetric Toeplitz matrices with a high level of accuracy, with a relatively low computational cost, and with potential application to the computation of the spectrum of differential operators. In fact, up to low rank corrections, matrices of the form $\mathcal{P}_n(f, g)$ appear in the context of the spectral approximation of differential operators in which a low rank correction of $T_n(g)$ is the mass matrix and a low rank correction of $T_n(f)$ is the stiffness matrix. We carried out also preliminary numerical tests confirming that the same kind of asymptotic expansion holds, at least in the context of the Isogeometric approximation of second-order differential operators.

Therefore, a plan for the future has to include:

- the theoretical proof of the asymptotic expansion in (1) for $\alpha > 1$;
- the analysis of the non monotone case and its relations with the study in [12] for the special case where $f(\theta) = 2 - 2 \cos(\omega\theta)$, $\omega \geq 2$ integer, and $g(\theta) = 1$;
- the extension of the results by [1] to the preconditioned Toeplitz case and the study of its connection with the general expansion in (1);
- the extension of the numerical and theoretical study to a multidimensional, block setting, with special attention to the matrices coming from the approximation of elliptic differential operators.

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with $\text{rank}(H_n(\psi)) \leq 2(m - 1)$.

Hence,

$$\begin{aligned} R_f &:= \text{rank}(H_n(f)) \leq 2(\text{deg}(f) - 1), \\ R_g &:= \text{rank}(H_n(g)) \leq 2(\text{deg}(g) - 1), \\ R_{f,g} &:= \max\{R_f, R_g\} \leq 2(\max\{\text{deg}(f), \text{deg}(g)\} - 1). \end{aligned} \tag{21}$$

Let P_n^τ be the matrix $\tau_n^{-1}(g)\tau_n(f)$,

$$\begin{aligned} P_n^\tau &= Q \left(\text{diag}_{1 \leq j \leq n} \left(g \left(\frac{j\pi}{n+1} \right) \right) \right)^{-1} Q Q \text{diag}_{1 \leq j \leq n} \left(f \left(\frac{j\pi}{n+1} \right) \right) Q \\ &= Q \text{diag}_{1 \leq j \leq n} \left(\frac{f}{g} \left(\frac{j\pi}{n+1} \right) \right) Q \\ &= Q \text{diag}_{1 \leq j \leq n} \left(r \left(\frac{j\pi}{n+1} \right) \right) Q. \end{aligned}$$

Hence, for $j = 1, \dots, n$

$$\lambda_j(P_n^\tau) = r \left(\frac{j\pi}{n+1} \right). \tag{22}$$

By observing that $T_n^{-1}(g)T_n(f)$ is similar to $T_n^{-1/2}(g)T_n(f)T_n^{-1/2}(g)$, using the MinMax spectral characterization for Hermitian matrices [2], fixed $j \in \{R_{f,g} + 1, \dots, n - R_{f,g}\}$ and $T \subset \mathbb{C}^n$, $\dim(T) = n + 1 - j$, we obtain

$$\begin{aligned} \lambda_j(P_n(f, g)) &= \lambda_j(T_n^{-1}(g)T_n(f)) \\ &= \lambda_j(T_n^{-1/2}(g)T_n(f)T_n^{-1/2}(g)) \\ &= \max_{\dim(T)=n+1-j} \left(\min_{\substack{x \in T, \\ x \neq \underline{0}}} \left(\frac{x^* T_n^{-1/2}(g) T_n(f) T_n^{-1/2}(g) x}{x^* x} \right) \right) \\ &= \max_{\dim(T)=n+1-j} \left(\min_{\substack{x \in T, \\ x \neq \underline{0}, \\ y = T_n^{-1/2}(g)x}} \left(\frac{y^* T_n(f) y}{y^* T_n(g) y} \right) \right) \\ &= \max_{\dim(\hat{T})=n+1-j} \left(\min_{\substack{y \in \hat{T}, \\ y \neq \underline{0}}} \left(\frac{y^* T_n(f) y}{y^* T_n(g) y} \right) \right), \end{aligned} \tag{23}$$

because $T_n^{-1/2}(g)$ is a full rank matrix and, if $\dim(T) = n + 1 - j$, then $\hat{T} := \{y : y = T_n^{-1/2}(g)x, x \neq \underline{0}, x \in T\}$ is a new vector space having the same dimension $n + 1 - j$ as T .

Let F be the subspace of \mathbb{C}^n generated by the union of the columns of matrices $H_n(f)$ and $H_n(g)$. Because of the particular structure of the columns of Hankel matrices $H_n(f)$ and $H_n(g)$, we deduce

$$\dim(F) = \max\{\text{rank}(H_n(g)), \text{rank}(H_n(f))\} = R_{f,g},$$

so that

$$\dim(F^\perp) = n - R_{f,g}.$$

Let us define $W_{f,g} = \hat{T} \cap F^\perp$,

$$\begin{aligned} n + 1 - j &\geq \dim(W_{f,g}) \geq \max\{0, \dim(\hat{T}) + \dim(F^\perp) - n\} = n + 1 - j \\ &\quad + n - R_{f,g} - n = n + 1 - (j + R_{f,g}), \end{aligned}$$

because $n + 1 - (j + R_{f,g}) \geq 1$ for $j \leq n - R_{f,g}$. The latter implies in particular that $W_{f,g} \neq \emptyset$. Thus, due to the orthogonality, $\forall y \neq \underline{0} \in W_{f,g}$, we find

$$H_n(f)y = \underline{0}, \quad H_n(g)y = \underline{0},$$

so that

$$y^* H_n(f)y = 0, \quad y^* H_n(g)y = 0.$$

Hence, from (23)

$$\begin{aligned} \lambda_j(\mathcal{P}_n(f, g)) &= \max_{\substack{\dim(\hat{T})=n+1-j \\ y \neq \underline{0}}} \left(\min_{y \in \hat{T}} \left(\frac{y^*(\tau_n(f) + H_n(f))y}{y^*(\tau_n(g) + H_n(g))y} \right) \right) \\ &\leq \max_{\dim(\hat{T})=n+1-j} \left(\min_{\substack{y \in W_{f,g} \\ y \neq \underline{0}}} \left(\frac{y^*(\tau_n(f) + H_n(f))y}{y^*(\tau_n(g) + H_n(g))y} \right) \right) \\ &= \max_{\dim(\hat{T})=n+1-j} \left(\min_{\substack{y \in W_{f,g} \\ y \neq \underline{0}}} \left(\frac{y^* \tau_n(f)y}{y^* \tau_n(g)y} \right) \right) \\ &= \max_{\substack{W_{f,g} = \hat{T} \cap F^\perp \\ \dim(\hat{T})=n+1-j}} \left(\min_{\substack{y \in W_{f,g} \\ y \neq \underline{0}}} \left(\frac{y^* \tau_n(f)y}{y^* \tau_n(g)y} \right) \right) \\ &\leq \max_{n+1-j \geq \dim(\hat{W}_{f,g}) \geq n+1-(j+R_{f,g})} \left(\min_{\substack{y \in \hat{W}_{f,g} \\ y \neq \underline{0}}} \left(\frac{y^* \tau_n(f)y}{y^* \tau_n(g)y} \right) \right) \\ &= \max_{n+1-j \geq \dim(\hat{W}) \geq n+1-(j+R_{f,g})} \left(\min_{\substack{y \in \hat{W}_{f,g} \\ y \neq \underline{0} \\ x = \tau_n^{-1/2}(g)y}} \left(\frac{x^* \tau_n^{-1/2}(g) \tau_n(f) \tau_n^{-1/2}(g)x}{x^* x} \right) \right) \\ &= \max\{\lambda_j(P_n^\tau), \lambda_{j+1}(P_n^\tau), \dots, \lambda_{j+R_{f,g}}(P_n^\tau)\} \\ &= \lambda_{j+R_{f,g}}(P_n^\tau). \end{aligned} \tag{24}$$

By fixing $j \in \{R_{f,g} + 1, \dots, n - R_{f,g}\}$ and $T \subset \mathbb{C}^n$, $\dim(T) = j$, analogously we obtain

$$\begin{aligned} \lambda_j(\mathcal{P}_n(f, g)) &= \min_{\dim(T)=j} \left(\max_{\substack{x \in T \\ x \neq \underline{0}}} \left(\frac{x^* T_n^{-1/2}(g) T_n(f) T_n^{-1/2}(g)x}{x^* x} \right) \right) \\ &= \min_{\dim(T)=j} \left(\max_{\substack{x \in T \\ x \neq \underline{0} \\ y = T_n^{-1/2}(g)x}} \left(\frac{y^* T_n(f)y}{y^* T_n(g)y} \right) \right) \\ &= \min_{\dim(\hat{T})=j} \left(\max_{\substack{y \in \hat{T} \\ y \neq \underline{0}}} \left(\frac{y^* T_n(f)y}{y^* T_n(g)y} \right) \right) \\ &= \min_{\dim(\hat{T})=j} \left(\max_{\substack{y \in \hat{T} \\ y \neq \underline{0}}} \left(\frac{y^*(\tau_n(f) + H_n(f))y}{y^*(\tau_n(g) + H_n(g))y} \right) \right). \end{aligned} \tag{25}$$

Let us define $W_{f,g} = \hat{T} \cap F^\perp$,

$$j \geq \dim(W_{f,g}) \geq \max\{0, \dim(\hat{T}) + \dim(F^\perp) - n\} = j + n - R_{f,g} - n = j - R_{f,g},$$

because $j - R_{f,g} \geq 1$ for $j \geq R_{f,g} + 1$. The latter implies in particular that $W_{f,g} \neq \emptyset$, and hence, due to the orthogonality, $\forall y \neq \underline{0} \in W_{f,g}$, we have

$$H_n(f)y = \underline{0}, \quad H_n(g)y = \underline{0},$$

and therefore

$$y^* H_n(f)y = 0, \quad y^* H_n(g)y = 0.$$

Thus, from (25)

$$\begin{aligned} \lambda_j(\mathcal{P}_n(f, g)) &\geq \min_{\dim(\hat{T})=j} \left(\max_{\substack{y \in W_{f,g} \\ y \neq 0}} \left(\frac{y^*(\tau_n(f) + H_n(f))y}{y^*(\tau_n(g) + H_n(g))y} \right) \right) \\ &= \min_{\dim(\hat{T})=j} \left(\max_{\substack{y \in W_{f,g} \\ y \neq 0}} \left(\frac{y^* \tau_n(f)y}{y^* \tau_n(g)y} \right) \right) \\ &= \min_{\substack{W_{f,g} = \hat{T} \cap F^\perp \\ \dim(\hat{T})=j}} \left(\max_{\substack{y \in W_{f,g} \\ x \neq 0}} \left(\frac{y^* \tau_n(f)y}{y^* \tau_n(g)y} \right) \right) \tag{26} \\ &\geq \min_{j \geq \dim(\hat{W}_{f,g}) \geq j - R_{f,g}} \left(\max_{\substack{y \in W_{f,g} \\ y \neq 0}} \left(\frac{y^* \tau_n(f)y}{y^* \tau_n(g)y} \right) \right) \\ &= \min\{\lambda_j(P_n^\tau), \lambda_{j-1}(P_n^\tau), \dots, \lambda_{j-R_{f,g}}(P_n^\tau)\} \\ &= \lambda_{j-R_{f,g}}(P_n^\tau). \end{aligned}$$

By exploiting the previous inequality, relations (22) and (24), we obtain for $j = R_{f,g} + 1, \dots, n - R_{f,g}$

$$r \left(\frac{(j - s)\pi}{n + 1} \right) = \lambda_{j-s}(P_n^\tau) \leq \lambda_j(\mathcal{P}_n(f, g)) \leq \lambda_{j+s}(P_n^\tau) = r \left(\frac{(j + s)\pi}{n + 1} \right), \tag{27}$$

where $s = R_{f,g}$.

The function r is a RCTP on $[0, \pi]$ and a monotone increasing function so we have, $\forall n$ and $\forall j = s + 1, \dots, n - s$,

$$\lambda_j(\mathcal{P}_n(f, g)) - r \left(\frac{j\pi}{n + 1} \right) \leq r \left(\frac{(j + s)\pi}{n + 1} \right) - r \left(\frac{j\pi}{n + 1} \right) = r'(\bar{\theta}) \frac{s\pi}{n + 1} \leq \|r'\|_\infty s\pi h, \tag{28}$$

with $\bar{\theta} \in \left(\frac{j\pi}{n + 1}, \frac{(j + s)\pi}{n + 1} \right)$ and

$$\lambda_j(\mathcal{P}_n(f, g)) - r \left(\frac{j\pi}{n + 1} \right) \geq r \left(\frac{(j - s)\pi}{n + 1} \right) - r \left(\frac{j\pi}{n + 1} \right) \geq -\|r'\|_\infty s\pi h. \tag{29}$$

By setting $C = \|r'\|_\infty s\pi$, for $s + 1 \leq j \leq n - s$, we obtain

$$\left| \lambda_j(\mathcal{P}_n(f, g)) - r \left(\frac{j\pi}{n + 1} \right) \right| \leq Ch. \tag{30}$$

Furthermore, from [11] $\forall j = 1, \dots, n$, we know that

$$m_r \leq \lambda_j(\mathcal{P}_n(f, g)) \leq M_r,$$

where

$$m_r = \min_{\theta \in [0, \pi]} r(\theta); \quad M_r = \max_{\theta \in [0, \pi]} r(\theta),$$

with strict inequalities that is $m_r < \lambda_j(\mathcal{P}_n(f, g)) < M_r$ if $m_r < M_r$, while the case $m_r = M_r$ is in fact trivial. Hence, for $n - s < j \leq n$

$$\left| r \left(\frac{j\pi}{n + 1} \right) - \lambda_j(\mathcal{P}_n(f, g)) \right| \leq \left| r \left(\frac{j\pi}{n + 1} \right) - r \left(\frac{n\pi}{n + 1} \right) \right| \leq |r'(\bar{\theta})| \left| \frac{(n - j)\pi}{n + 1} \right|,$$

where $\bar{\theta} \in (\frac{j\pi}{n+1}, \frac{n\pi}{n+1})$. If $n - s < j \leq n$ then $|n - j| < s$, so that

$$\left| r \left(\frac{j\pi}{n+1} \right) - \lambda_j(\mathcal{P}_n(f, g)) \right| \leq \|r'\|_\infty s\pi h = Ch.$$

For $1 \leq j < s + 1$

$$\left| r \left(\frac{j\pi}{n+1} \right) - \lambda_j(\mathcal{P}_n(f, g)) \right| \leq \left| r \left(\frac{j\pi}{n+1} \right) - r \left(\frac{\pi}{n+1} \right) \right| \leq |r'(\bar{\theta})| \left| \frac{(j-1)\pi}{n+1} \right|,$$

where $\bar{\theta} \in (\frac{\pi}{n+1}, \frac{j\pi}{n+1})$. If $1 \leq j < s + 1$ then $|j - 1| < s$, so

$$\left| r \left(\frac{j\pi}{n+1} \right) - \lambda_j(\mathcal{P}_n(f, g)) \right| \leq \|r'\|_\infty s\pi h = Ch.$$

Hence,

$$\left| \lambda_j(\mathcal{P}_n(f, g)) - r \left(\frac{j\pi}{n+1} \right) \right| \leq Ch \quad \forall j \forall n.$$

□

Here, we present a second proof of the previous theorem.

Proof We adopt the very same notation used for the first proof. First, we notice that the low rank matrices $H_n(f)$ and $H_n(g)$ are also Hermitian matrices because $T_n(f)$, $T_n(g)$, $\tau_n(f)$, and $\tau_n(g)$ are Hermitian matrices. Let \mathbf{x}_i and $\lambda_i(\mathcal{P}_n(f, g))$ be a pair eigenvector and eigenvalue of $\mathcal{P}_n(f, g)$. Then we can write

$$\mathcal{P}_n(f, g)\mathbf{x}_i = \lambda_i(\mathcal{P}_n(f, g))\mathbf{x}_i.$$

By multiplying the previous equation from the left by the matrix $T_n(g) = \tau_n(g) + H_n(g)$, we obtain

$$(\tau_n(f) + H_n(f))\mathbf{x}_i = \lambda_i(\mathcal{P}_n(f, g))(\tau_n(g) + H_n(g))\mathbf{x}_i,$$

which is equivalent to

$$(\tau_n(f) + H_n(f) - \lambda_i(\mathcal{P}_n(f, g))H_n(g))\mathbf{x}_i = \lambda_i(\mathcal{P}_n(f, g))\tau_n(g)\mathbf{x}_i.$$

Finally, by setting $\mathbf{y}_i = \tau_n^{-1/2}(g)\mathbf{x}_i$ and by multiplying from the left by the matrix $\tau_n^{-1/2}(g)$, we have

$$\tau^{-1/2}(g)(\tau_n(f) + H_n(f) - \lambda_i(\mathcal{P}_n(f, g))H_n(g))\tau^{-1/2}(g)\mathbf{y}_i = \lambda_i(\mathcal{P}_n(f, g))\mathbf{y}_i. \tag{31}$$

Equation (31) tells us that $\lambda_i(\mathcal{P}_n(f, g))$ is also the eigenvalue of

$$\tau_n^{-1/2}(g)(\tau_n(f) + H_n(f) - \lambda_i(\mathcal{P}_n(f, g))H_n(g))\tau_n^{-1/2}(g).$$

We can write

$$\tau_n^{-1/2}(g)(\tau_n(f) + H_n(f) - \lambda_i(\mathcal{P}_n(f, g))H_n(g))\tau_n^{-1/2}(g)$$

as

$$\begin{aligned} & \tau_n^{-1/2}(g)\tau_n(f)\tau_n^{-1/2}(g) + \tau_n^{-1/2}(g)(H_n(f) - \lambda_i(\mathcal{P}_n(f, g))H_n(g))\tau_n^{-1/2}(g) \\ & = \tau_n(f/g) + \tau_n^{-1/2}(g)(H_n(f) - \lambda_i(\mathcal{P}_n(f, g))H_n(g))\tau_n^{-1/2}(g). \end{aligned} \tag{32}$$

Notice that the rank of any linear combination of $H_n(f)$ and $H_n(g)$ is $R_{f,g} = \max\{\text{rank}(H_n(f)), \text{rank}(H_n(g))\}$ and the argument is the special Hankel structure of $H_n(f)$ and $H_n(g)$. As a conclusion, from the expression above, using the MinMax characterization and the interlacing theorem for Hermitian matrices, we write

$$\lambda_{i-R_{f,g}}(\tau_n(f/g)) \leq \lambda_i(\mathcal{P}_n(f, g)) \leq \lambda_{i+R_{f,g}}(\tau_n(f/g)), \quad (33)$$

where $i \in \{R_{f,g}-1, \dots, n-R_{f,g}\}$, which leads again to the proof of Theorem 1. \square

Remark With regard to Theorem 1, the case where r is bounded and nonmonotone is even easier. If we consider \hat{r} , the monotone nondecreasing rearrangement of r on $[0, \pi]$, taking into account that the derivative of r has at most a finite number S of sign changes, we deduce that \hat{r} is Lipschitz continuous and its Lipschitz constant is bounded by $\|r'\|_\infty$ (notice that \hat{r} is not necessarily continuously differentiable, but the derivative of \hat{r} has at most S points of discontinuity). Furthermore, the eigenvalues of $\tau_n(r)$ are exactly given

$$r\left(\frac{j\pi}{n+1}\right)$$

so that, by ordering these values nondecreasingly, we deduce that they coincide with $\hat{r}(x_{j,h})$, with $x_{j,h}$ of the form $\frac{j\pi}{n+1}(1+o(1))$. With these premises, the proof follows exactly the same steps as in Theorem 1, using the MinMax characterization and the interlacing theorem for Hermitian matrices.

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