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REPLY TO LETTER TO THE EDITOR

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Eigenpairs of some particular band Toeplitz matrices: A comment

REPLY FROM AUTHORS

We thank C. M. da Fonseca for his interesting and valuable comments on the paper.[1] The authors, S.-E. Ekström and S. Serra-Capizzano, acknowledge that the comments are pertinent and we admit that we were previously unaware of part of the works referenced in the **Letter to the Editor** (Eigenpairs of some particular band Toeplitz matrices: A comment) by C. M. da Fonseca.

The proposed references by C. M. da Fonseca should be recognized as previous similar results as in Section 2 of the work of Ekström et al.,[1] which were developed independently. On the other hand, the journal where we published our contribution has the target of numerical methods in the context of linear algebra, and indeed, the theoretical results considered by C. M. da Fonseca are only an intermediate step for proposing numerical methods in a more general setting. Furthermore, we note that the approach of defining special sampling grids for the spectral symbol is conceptually a different interpretation of the problem (for example, C. M. da Fonseca mentions multivariable operator theory or graph theory[2]), while we aim at numerical algorithms in the more involved case of general banded Toeplitz structures. In fact, our approach leads to an important novel part of the work of Ekström et al.,[1] namely, the discussions of Section 3.

Regarding Section 3 of the work of Ekström et al.,[1] where the importance of these presented results in the scope of Toeplitz matrices generated by *nonmonotone spectral symbols* and the use of the *asymptotic expansion of eigenvalues* (see other works,[3–6] and references therein) are discussed, we here highlight few relevant results stemming from the work of Ekström et al.,[1] developed since publication:

- In Section 2.5 of the work of Ekström,[4] a second view point of the eigenvalue distribution, with modified versions of the formulae (Theorems 1 and 2) of Ekström et al.,[1] based on block diagonalization is presented. In addition, Ekström[4] further discussed the approach of multiple sampling grids to attain better approximations of the eigenvalues when using the spectral symbol.
- The results in the previous item have been improved, and preliminary results exist using the idea of multiple grids on different parts of the domain, for approximating the asymptotic expansion (see other works,[3–6] and references therein) for some nonmonotone spectral symbols.
- A similar approach, as in the work of Ekström et al.,[1] has been extended to what might be called (k_1, k_2)-tridiagonal (or (k, l)-tridiagonal, or nonsymmetrically sparse tridiagonal) *Toeplitz matrices*, where k_1, k_2 are arbitrary positive integers (less than n), which is under preparation. The suggested references by C. M. da Fonseca will be considered in our upcoming works.

Regarding definitions, we think that our name *symmetrically sparse tridiagonal Toeplitz matrices* is appropriate to characterize the discussed matrices in the work of Ekström et al.[1] On the other hand, the term *k-tridiagonal Toeplitz matrices*[7,8] suggested by C. M. da Fonseca for the problems in the work of Ekström et al.[1] is misleading because the sets of the considered matrices do not coincide. Both classes are subclasses of (truncated) Toeplitz matrices $T_N(f)$ having a $k \times k$ matrix-valued generating function of the form

$$f(\theta) = A_0 + A_1 \mathrm{e}^{\mathrm{i}\theta} + A_{-1}\mathrm{e}^{-\mathrm{i}\theta}, \tag{1}$$

$$T_N(f) = \begin{bmatrix} A_0 & A_{-1} & & & \\ A_1 & \ddots & \ddots & & \\ & \ddots & & \ddots & \\ & & \ddots & \ddots & A_{-1} \\ & & & A_1 & A_0 \end{bmatrix}.$$

In the case of *k-tridiagonal Toeplitz matrices*, matrix A_0 is a generic tridiagonal matrix, matrix A_1 is a dyad of the form $\alpha e_1 e_k^T$, and matrix A_{-1} is a dyad of the form $\beta e_k e_1^T$, e_j being the jth vector of the canonical basis of complex vectors in k dimensions. In the case of *symmetrically sparse tridiagonal Toeplitz matrices*, A_0, A_1, A_{-1} are three generic multiples of the identity. Hence, *k-tridiagonal Toeplitz matrices* are not in general *symmetrically sparse tridiagonal Toeplitz matrices* and vice versa, but both are (truncated) Toeplitz matrices $T_N(f)$ having a $k \times k$ matrix-valued generating function of the form in (1). Furthermore, looking at *k-tridiagonal Toeplitz matrices* as truncated Toeplitz matrices is fruitful for analyzing, e.g., the distribution of the zeros of orthogonal polynomials with asymptotically periodic coefficients, also in the varying coefficient setting[9] (see also the of work of Kuijlaars et al.[10] for the use of GLT results[11] in the context of zero distribution of orthogonal polynomials and see the works of Golinskii et al.[12,13] for further related applications).

Finally, looking at the papers of da Fonseca et al.,[8,14] we observe that huge literature on the inverses of tridiagonal and banded structures is not considered. We recommend the paper of Vandebril et al.[15] on semiseparable matrices, where among other more sophisticate results, a substantially complete history on the discovery and rediscovery of formulas for inverses of tridiagonal matrices is given. Here, we report only the first 10 contributions in chronological order essentially quoting from the work of Vandebril et al.[15]:

- 1937: paper by F. R. Gantmacher and M. G. Krein on the inverse of a symmetric Jacobi matrix via explicit calculations;
- 1941: book of F. R. Gantmacher and M. G. Krein (reporting the analysis in their 1937 paper);
- 1953: W.J. Berger and E. Saibel provide in this paper an explicit formula for calculating the inverse of a continuant matrix (a tridiagonal matrix, not necessarily symmetric);
- 1956: the authors S. N. Roy and A. E. Sarhan invert very specific matrices arising in statistical applications;
- 1959: in this paper, S. O. Asplund, the father of E. Asplund, proves the result of F. R. Gantmacher and M. G. Krein;
- 1959: E. Asplund formulates for the first time a theorem regarding the inverse of an invertible lower p-Hessenberg matrix;
- 1959: in this paper, by B. G. Greenberg and A. E. Sarhan, there are several generalizations concerning several types of matrices arising in statistical applications;
- 1960: as the statistical research at that time was interested in fast calculations for so-called patterned matrices, the authors S. N. Roy, B. G. Greenberg, and A. E. Sarhan designed an order n algorithm for calculating the determinant. The patterned matrices are semiseparable and semiseparable plus diagonal matrices;
- 1960: M. Schechter provides a method for inverting nonsymmetric block tridiagonal matrices for blocks, which have the same size and are invertible;
- 1963: G. H. Golub investigates the optimal scaling of certain variance matrices, which are of semiseparable form, based on the scaling of their inverses.

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