Can We Compute the Eigenvalues Without the Matrix?

A new type of method, called "**matrix-less**" [1], can be used to approximate the eigenvalues of a wide variety of matrices. The name comes from the fact that a matrix does not have to be constructed to approximate its eigenvalues.

These methods are based on the so-called of generalized locally Toeplitz (**GLT**) sequences [2]. If we are interested in the eigenvalues, $\lambda_j(A_N)$, of a matrix $A_N \in \mathbb{C}^{N\times N}$, we can associate a function $f(\theta)$, called the **symbol**, to a sequence of matrices $\{A_n\}_n$ of which A_N belongs. By defining a grid, for example,

$$\theta_j = \frac{j\pi}{N+1}, \quad j = 1, \dots, N,$$

we can then approximate the eigenvalues of A_N by sampling the symbol $f(\theta)$ with the grid θ_j,

$$\lambda_j(A_N) = f(\theta_j) + E_j,$$

where E_j is the error for each eigenvalue approximation. These errors are typically $\mathcal{O}(h)$, where $h = 1/(N+1)$. In the matrix-less method we construct a "higher order symbol", $f_\alpha(\theta)$, such that,

$$\lambda_j(A_N) = f_\alpha(\theta_j) + E_{j,\alpha},$$

where the errors $E_{j,\alpha} = \mathcal{O}(h^{\alpha+1})$; this is much more accurate than the approximation given by the symbol $f(\theta)$.

Since these methods are very new, and still under active research and development, there are still many types of problems to apply them to.

In this project you will investigate the behavior of a matrix-less method for **problems with variable coefficients**. That is, can we describe the behavior of the eigenvalues of the discretization matrices from, e.g.,

$$(a(x)u_x)_x \tag{1}$$

using standard central finite differences (FDM)? Here, $a(x)$ can be a continuous/discontinuous and monotone/nonmonotone function.

This project relies heavily on **experimental mathematics** to gain insights on pure linear algebra topics, and you will

- learn the basics of the theory of GLT sequences and matrix-less methods,
- develop numerical experimental tools,
- devise and execute experiments,
- and investigate real research problems.

Planned Tasks (many extensions are possible)

1. Understand the matrix-less method to be used (working code given in Julia [3] or Matlab).
2. Implement a FDM discretization of the problem (1).
3. Study and describe the behavior of the eigenvalues and the method for different $a(x)$.

Practical details

- Supervisor: Sven-Erik Ekström, PostDoc at the division of scientific computing, IT department, Uppsala University, `sven-erik.ekstrom@it.uu.se` (`www.2pi.se`)
- Prerequisites: Basic understanding of linear algebra and programming.
- Prefered programming language is Julia [3], no experience required. Any language can be used.
- Meetings and discussions to be carried out on platforms like IRC, Slack, Discord, and Zoom.

References

[1] S.-E. Ekström, *Matrix-Less Methods for Computing Eigenvalues of Large Structured Matrices*, Ph.D. Thesis, Uppsala University (2018) (`www.2pi.se/thesis.pdf`)
[2] C. Garoni and S. Serra-Capizzano, *Generalized Locally Toeplitz Sequences: Theory and Applications*, Springer, 2017 (`www.doi.org/10.1007/978-3-319-53679-8`)
[3] J. Bezanson, A. Edelman, S. Karpinski, and V. Shah, *Julia: A fresh approach to numerical computing*, SIAM review 59:1, pp. 65–98 (2017) (`www.julialang.org`)